Chapter 13

General Probability

A Question to Start

A disease affects 1 in 1,000 people. A test for the disease is 99% accurate, meaning:

- If you *have* the disease, the test gives a positive result 99% of the time.
- If you *don't* have the disease, the test gives a negative result 99% of the time.

You take the test and receive a **positive** result.

Find: What is the probability you actually have the disease?

Most People Guess Wrong

When researchers pose this question, the most common answers fall between 90% and 99%.

Common guesses: 90–99%

Actual answer:

This is not about being bad at math.

Human intuition struggles with conditional probability when base rates are small.

Why Does This Happen?

The test is accurate, but the disease is rare. When you test positive, there are two possible explanations:

1 in 1,000

True positive
You have the disease and
the test correctly detected it.

999 in 1,000

False positive
You do not have the disease but
the test incorrectly flagged you.

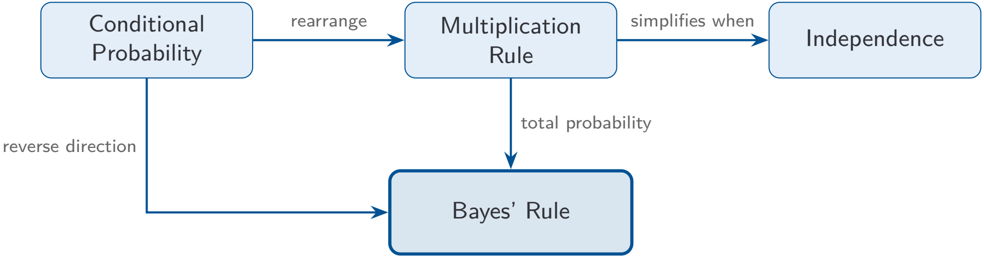
Conditional probability gives us the tools to figure out which case is more likely. Let's begin.

Intended Learning Outcomes

- Define and compute conditional probabilities
- Apply the multiplication rule
- Use tree diagrams for multi-stage problems
- Define and test for independence
- Use the Law of Total Probability
- Apply Bayes' Rule to reverse conditionals
- Distinguish independence from mutual exclusivity
- Avoid common probability reasoning errors

Chapter 13: The Big Picture

How does learning new information change probabilities?



PART 1

Conditional Probability

How does knowing B occurred change the probability of A?

Updating Probability with New Information

How does learning new information change what we believe?

- A patient tests positive for a disease. What is the probability they actually have it?
- A student is in first year. What is the probability they live on campus?
- You rolled an even number. What is the probability it was a 2?

Key Point: Conditional probability answers: "What is the probability of A, *given that* we know B has occurred?"

Conditional Probability: Definition

Conditional Probability

The **conditional probability** of event A given event B has occurred is:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

provided that $P(B) > 0$.

- The bar "$|$" is read "given"
- Denominator $P(B)$ restricts to outcomes where B occurred
- Conditioning "zooms in" on the subset of outcomes where B happened

Conditional Probability: Terminology

Joint and Marginal Probabilities

$P(A \text{ and } B)$: **joint probability** — the probability both events occur.

$P(A)$, $P(B)$ alone: **marginal probabilities** — the probability of a single event.

Conditioning on B means we zoom in on outcomes where B occurred and ask what fraction are also in A.

Conditional Probability with a Die

Example 13.1

Roll a fair six-sided die. Let $A =$ "roll a 2" and $B =$ "roll is even."

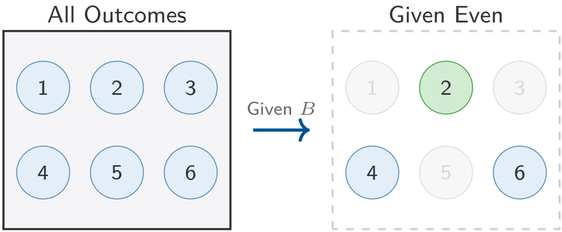
Calculate: The conditional probabilities below.

(a) Find $P(A \mid B)$:

(b) Find $P(B \mid A)$:

The Key Insight: Restricting the Sample Space

Key Point: Conditional probability *restricts* our attention to only those outcomes where the condition occurred.



Conditional Probability from a Table

Example 13.2

Context: The NCAA surveyed 3,527 young adults (ages 18–22) about sports betting.
Calculate: Marginal and conditional probabilities from the table.

	Bets	**Does Not**	**Total**
Male	1,182	582	1,764
Female	899	864	1,763
Total	2,081	1,446	3,527

(a) $P(\text{Bets on sports})$

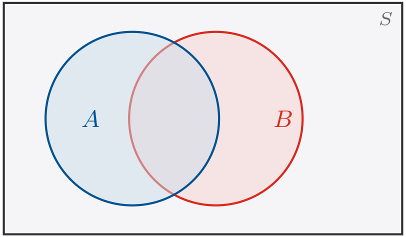
(b) $P(\text{Bets} \mid \text{Male})$

(c) $P(\text{Female} \mid \text{Does Not Bet})$

Visualizing Conditional Probability

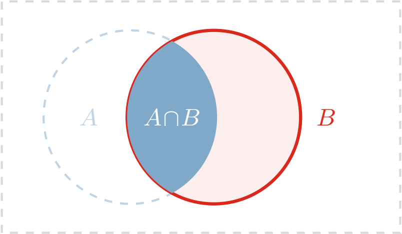
Example 13.3

Marginal: $P(A)$



$$P(A) = \frac{|A|}{|S|}$$

Conditional: $P(A \mid B)$



$$P(A \mid B) = \frac{|A \cap B|}{|B|}$$

Key Point: Conditioning on B replaces S with B. What fraction of B lies in A?

PART 2

The Multiplication Rule

Finding joint probabilities from conditional ones

From Conditional to Joint Probability

If we know $P(A \mid B)$, can we find $P(A \text{ and } B)$?

Starting from the definition and rearranging:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Multiplication Rule

For any events A and B:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

$P(A \text{ and } B)$ equals $P(B)$ times the probability that A also occurs given B.

Applying the Multiplication Rule

Example 13.5

Context: A 2023 NCAA survey found that betting rates differ by gender among 18–22 year olds. We want to find the probability that a randomly selected person is both male and bets on sports.

- 67% of males bet on sports
- 51% of females bet on sports
- Sample: 50% male, 50% female

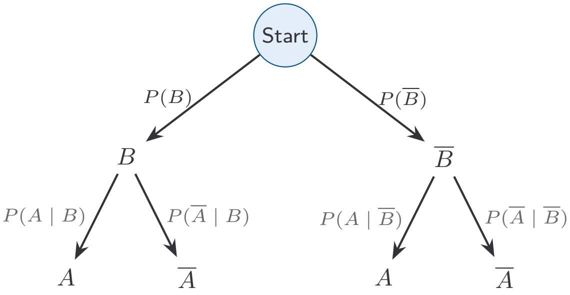
Find: $P(\text{male and bets on sports})$

Tree Diagrams: A Visual Tool for Multi-Stage Problems

Tree Diagram

A **tree diagram** displays all possible outcomes of a sequence of events, with each branch showing a conditional probability.

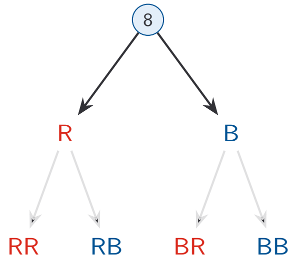
Key Point: To find the probability of a path:
Multiply the probabilities along the branches.



Tree Diagram for Drawing Without Replacement

Example 13.6

Context: A box has 5 red and 3 blue marbles. You draw two marbles one at a time without putting the first one back.
Calculate: Use the tree diagram to find the probabilities below.



(a) $P(\text{both red})$

(b) $P(\text{2nd blue} \mid \text{1st red})$

(c) $P(\text{exactly one red})$

Multiplication Rule for Three or More Events

Example 13.7 — Setup

Extended Multiplication Rule

For events A, B, and C:

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B)$$

Context: Based on Western University's Schulich School of Medicine admissions process. Three sequential stages must all be passed:

- Screening: 45% of applicants pass
- Interview: 40% of those screened pass
- Final Review: 30% of those interviewed are admitted

Find: What is the probability that a random applicant makes it through all three stages?

Multiplication Rule for Three or More Events

Example 13.7 — Calculation

Context: Screening (45%), Interview (40%), Final Review (30%).

Calculate: P(applicant passes all three stages)

PART 3

Independence

When does knowing B tell us nothing about A?

Independent Events: Definition

Independence

Two events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B)$$

- Knowing B occurred does not change the probability of A.
- The general multiplication rule simplifies: no conditional probability needed.

Multiplication Rule for Independent Events

Example 13.9

Roll a third die. Let a be the event of rolling a two, and let b be the event of rolling an even number. Determine if these two events are independent

If A and B are **independent**, and B is an event with non-zero probability, then

$$P(A \mid B) = P(A).$$

In other words, knowing that B occurred does not change the probability of A.

Testing for Independence

Example 13.10 — Setup

Context: Does having a meal plan make students more likely to eat breakfast? A survey of 200 university students found:

- 120 have a meal plan; of these, 90 eat breakfast daily.
- 80 do not have a meal plan; of these, 30 eat breakfast daily.

Test: Are having a meal plan (M) and eating breakfast daily (B) independent?

To test independence, check either:

- Is $P(B \mid M) = P(B)$?
- Or: Is $P(B \cap M) = P(B) \cdot P(M)$?

Testing for Independence

Example 13.10 — Calculation

Context: 200 students. Meal plan: 120 (90 eat breakfast). No meal plan: 80 (30 eat breakfast).

(a) $P(B)$

(b) $P(B \mid M)$

(c) Independent?

A Subtle Difference in Conditioning

Example 13.11

Context: A household adopts two dogs. Each dog is equally likely to be male (M) or female (F), independently.

The sample space is: $\{MM, MF, FM, FF\}$, each with probability $\frac{1}{4}$.

(a) $P(\text{both male} \mid \text{first is male})$

(b) $P(\text{both male} \mid \text{at least one male})$

Key Point: The answers differ because the conditions restrict the sample space differently. "The first is male" leaves 2 outcomes; "at least one is male" leaves 3.

The Gambler's Fallacy: Birth Sequences

Task: Which birth sequence is most likely for a family of 4 children?

- BBBB
- BGBG
- GGGG

The Gambler's Fallacy

At a roulette table, the last 7 spins landed on red. Should you bet on black?

Caution: The Gambler's Fallacy: Believing past independent events affect future probabilities.

Each spin is **independent**. The wheel has no memory. $P(\text{Black}) = \frac{18}{38}$ regardless of history.

PART 4

Law of Total Probability & Bayes' Rule

Reversing the direction of conditional probability

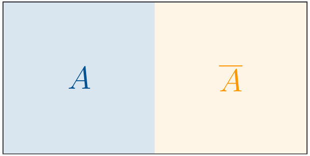
Partitioning the Sample Space

Partition

Events $A_1, A_2, \ldots, A_k$ form a **partition** of the sample space S if:

1. They are **mutually exclusive**: no two can occur at the same time.
2. They are **exhaustive**: together they cover all of S.

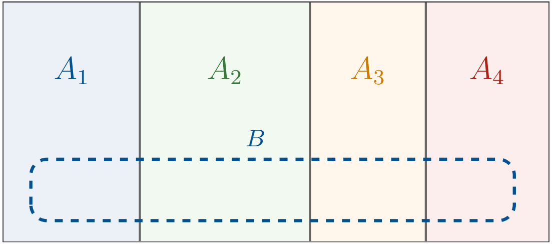
Simplest partition: A and $\overline{A}$



Key Point: Every outcome in S falls into exactly one piece of the partition. No gaps, no overlaps.

A More Complex Partition

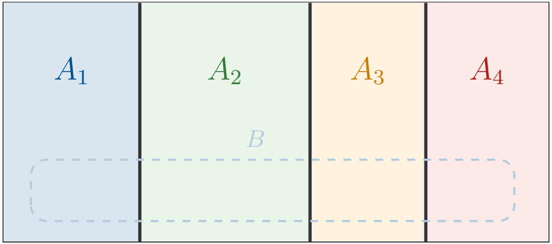
Events A_1, A_2, A_3, A_4 and event B



The events A_1, A_2, A_3, A_4 divide the sample space into regions. Event B can overlap with several of them.

The Partition: Mutually Exclusive and Exhaustive

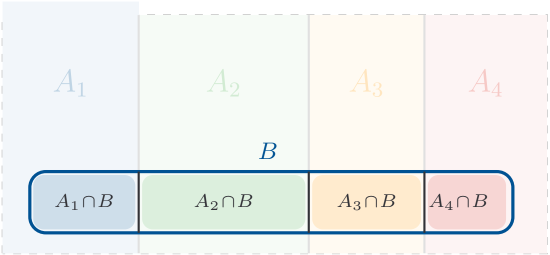
A_1, A_2, A_3, A_4 partition the sample space S



Key Point: The A_i are mutually exclusive (no overlaps) and exhaustive (no gaps). Every outcome in S belongs to exactly one A_i.

The Partition Also Splits B

A_1, A_2, A_3, A_4 also partition B



Key Point: The partition of S creates a partition of B: every outcome in B falls into exactly one $A_i \cap B$. This is the foundation of the Law of Total Probability.

Computing $P(B)$ from a Partition

Context: A university bookstore orders laptops from four suppliers. The table shows each supplier's share of orders and the probability that a laptop from that supplier is defective.

Supplier	**Share of Orders**	**Defect Rate**
TechCo	40%	2%
BuildRight	25%	5%
QuickChip	20%	4%
ValuePC	15%	10%

A laptop is selected at random. What is the probability it is defective?

Law of Total Probability

Law of Total Probability

If $A_1, A_2, \ldots, A_k$ partition S, then

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)\, P(A_i)$$

for any event B.

Applying the Law of Total Probability

Example 13.7 — Extended

Context: Returning to the bookstore laptop problem. A laptop is selected at random and found to be defective. Which supplier is it most likely from?

Recall:

Supplier	Share	Defect Rate
TechCo	40%	2%
BuildRight	25%	5%
QuickChip	20%	4%
ValuePC	15%	10%

Find: $P(\text{ValuePC} \mid \text{Defective})$

Reversing Conditional Probabilities

We often know $P(B \mid A)$ but need $P(A \mid B)$.

Context: A medical test for a rare disease:

- The test correctly detects 95% of people who *have* the disease.
- The test correctly clears 90% of people who *do not* have it.

You test positive. What is the probability you actually have the disease? We know $P(+ \mid D)$, but we need $P(D \mid +)$.

Key Point: Bayes' Rule lets us "flip" the direction of a conditional probability.

This is essential for:

- Medical diagnosis (COVID tests)?
- Quality control

Bayes' Rule

Bayes' Rule

For events A and B with $P(A) > 0$, $P(B) > 0$:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Where does $P(B)$ come from? Use the Law of Total Probability:

$$P(B) = P(B \mid A)\, P(A) + P(B \mid \overline{A})\, P(\overline{A})$$

Key Point: **Numerator:** Probability of pathway $A \to B$.
Denominator: Total probability of B from all pathways.
What fraction of all paths to B came through A?

COVID-19 Rapid Antigen Tests

Example 13.12 — Setup

Context: A rapid antigen test is used to screen for COVID-19. Based on a large study (117,372 samples):

- 5% of the population currently has COVID.
- The test correctly identifies 73% of people who have COVID (true positive rate).
- Among people who do *not* have COVID, 0.7% incorrectly test positive (false positive rate).

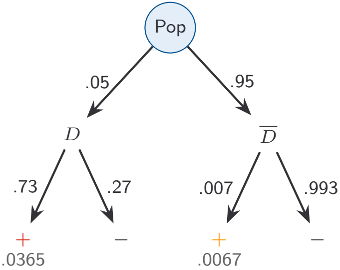
Find: A person tests positive. What is the probability they actually have COVID?

Caution: Even a test with a low false positive rate can produce many false positives when the disease is rare.

COVID-19 Rapid Antigen Tests

Example 13.12 — Tree Diagram

Context: 5% have COVID. The test correctly identifies 73% of those with COVID and incorrectly flags 0.7% of those without.



Two pathways lead to a positive test:

- Has COVID and tests positive:
  $0.05 \times 0.73 = 0.0365$
- No COVID but tests positive:
  $0.95 \times 0.007 = 0.0067$

Find: What fraction of all positive tests are true positives?

COVID-19 Rapid Antigen Tests

Example 13.12 — Calculation

Calculate: Use Bayes' Rule to find $P(\text{COVID} \mid +)$.

AI Writing Detection (Turnitin)

Example 13.13 — Setup

Context: Universities use Turnitin to flag student papers that may have been written by AI. A student's paper is flagged. Should the instructor assume it was AI-written?

Here is what we know about the detector's accuracy:

- It correctly flags 85% of papers that *are* AI-written.
- It incorrectly flags 1% of papers that are *human*-written.
- Suppose 10% of all submissions contain significant AI content.

Find: If a paper is flagged, what is the probability it was actually AI-written?

AI Writing Detection (Turnitin)

Example 13.13 — Calculation

Context: 85% detection rate, 1% false positive rate, 10% base rate of AI use.

Calculate: $P(\text{AI-written} \mid \text{flagged})$

When to Use Bayes' Rule

Key Point: Use Bayes' Rule when you need to **reverse** a conditional probability:

- You know $P(\text{result} \mid \text{condition})$
- You need $P(\text{condition} \mid \text{result})$

Bayes' Rule Steps

1. Identify: $P(B \mid A)$, $P(A)$
2. Need: $P(A \mid B)$
3. Find $P(B)$ via Total Probability
4. Apply: $P(A \mid B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

PART 5

Common Mistakes

Three errors to watch for in probability reasoning

Confusing $P(A \mid B)$ with $P(B \mid A)$

Common Mistake 1

Caution: The direction of the conditional matters. $P(A \mid B) \neq P(B \mid A)$ in general.

"The test is 95% accurate for people with the disease, so if I test positive, there is a 95% chance I have the disease."

This confuses $P(+ \mid D)$ with $P(D \mid +)$.

Quick example:

- $P(\text{wet} \mid \text{rain}) \neq P(\text{rain} \mid \text{wet})$
- $P(\text{cough} \mid \text{flu}) \neq P(\text{flu} \mid \text{cough})$
- $P(\text{bark} \mid \text{dog}) \neq P(\text{dog} \mid \text{bark})$

Key Point: In the COVID test example: $P(+ \mid D) = 73\%$ but $P(D \mid +) \approx 85\%$. These are different questions with different answers.

Assuming Independence Without Checking

Common Mistake 2

NCAA survey data on 3,527 young adults:

	Bets	**Not**	**Tot**
Male	1,182	582	1,764
Female	899	864	1,763
Total	2,081	1,446	3,527

Are gender (M) and betting (B) independent?

Confusing Mutually Exclusive with Independent

Common Mistake 3

Mutually Exclusive:

- Cannot both occur
- $P(A \cap B) = 0$
- Example: Roll 2 vs. roll 5

Independent:

- One does not affect the other
- $P(A \mid B) = P(A)$
- Example: Coin flip and die roll

Context: A university awards two scholarships, Athletics (A) and Arts (R), but a student cannot receive both.
$P(A) = 0.08$, $P(R) = 0.05$
Test: Are they independent?

Key Point: Mutually exclusive events (with non zero probability) are always **dependent**. Knowing B occurred means A did not: $P(A \mid B) = 0 \neq P(A)$.

Summary: Avoiding Common Probability Mistakes

1. Direction matters. $P(A \mid B) \neq P(B \mid A)$ in general.
   $P(\text{rain} \mid \text{clouds}) \neq P(\text{clouds} \mid \text{rain})$.

2. Do not assume independence. Always verify mathematically or from context.
   Events that seem unrelated may still be dependent.

3. Mutually exclusive $\neq$ independent. They are opposite concepts.
   Mutually exclusive events (with $P > 0$) are always dependent.

CHAPTER 13

Summary

Key concepts and formulas from this chapter

Chapter 13 Summary

Conditional Probability

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- Restricts sample space to B
- Direction matters.

Multiplication Rule

- $P(A \cap B) = P(B) \cdot P(A \mid B)$
- Extends to 3+ events
- Tree diagrams help visualize

Independence

- $P(A \mid B) = P(A)$
- If independent: $P(A \cap B) = P(A) \cdot P(B)$
- $\neq$ mutually exclusive.

Law of Total Probability

- $P(B) = \sum P(B \mid A_i)\, P(A_i)$
- Sums over all pathways to B

Bayes' Rule

- $P(A \mid B) = \frac{P(B|A) \cdot P(A)}{P(B)}$
- Reverses conditional direction

Looking Back and Looking Ahead

Key Point: Chapter 13 introduced tools for reasoning about related events: how knowing one thing changes the probability of another.

Key Skills:

- Compute conditional probabilities
- Use multiplication rule and trees
- Test for independence
- Apply Bayes' Rule

Real-World Applications:

- Medical testing (COVID-19 rapid tests)
- AI writing detection (Turnitin)
- Sports betting and risk assessment
- Airline on-time performance

PRACTICE

Problems

Test your understanding of conditional probability, independence, and Bayes' Rule

Practice Problem 1: Conditional Probability

Context: A survey of 500 employees at a company:

	Full-Time	**Part-Time**	**Total**
Remote	120	80	200
In-Office	210	90	300
Total	330	170	500

(a) Find $P(\text{Remote})$.
(b) Find $P(\text{Remote} \mid \text{Part-Time})$.
(c) Find $P(\text{Full-Time} \mid \text{Remote})$.
(d) Are working remotely and being full-time independent? Justify.

Practice Problem 2: Multiplication Rule and Trees

Context: A bag contains 4 red, 3 green, and 2 blue marbles. You draw two marbles without replacement.

(a) Draw a tree diagram for the first two draws (group by color).
(b) Find $P(\text{both marbles are red})$.
(c) Find $P(\text{second marble is green} \mid \text{first marble is red})$.
(d) Find $P(\text{at least one marble is blue})$.

Practice Problem 3: Independence

Context: At a gym, 60% of members use cardio equipment, 45% use weights, and 30% use both.

(a) Find $P(\text{weights} \mid \text{cardio})$.

(b) Find $P(\text{cardio} \mid \text{weights})$.

(c) Are using cardio and using weights independent events? Show your work.

(d) Are using cardio and using weights mutually exclusive? Explain.

Practice Problem 4: Bayes' Rule

Context: Three major U.S. airlines have the following market shares and on-time arrival rates (BTS/OAG, 2023):

- Delta: 35% of flights, 83% on-time
- United: 30% of flights, 79% on-time
- American: 35% of flights, 79% on-time

(a) If a random domestic flight is selected, what is the probability it arrives on time?
(b) If a flight is late, what is the probability it was a Delta flight?
(c) If a flight is late, which airline is it most likely from?

Practice Problem 5: Multi-Stage Process

Context: A hiring process has three stages:

- Resume screening: 40% of applicants pass
- Phone interview: 60% of those screened pass
- Final interview: 50% of those who pass phone interview receive offers

(a) What is the probability a random applicant receives an offer?

(b) If 1000 people apply, how many offers do you expect?

(c) Given that someone received an offer, what is the probability they passed all three stages? (Think carefully.)

Practice Problem 6: Conceptual Questions

For each statement, determine if it is TRUE or FALSE. Justify your answer.

(a) If $P(A) = 0.3$ and $P(B) = 0.4$, and A and B are independent, then $P(A \text{ or } B) = 0.7$.

(b) If $P(A \mid B) = 0$, then A and B must be mutually exclusive.

(c) If A and B are independent, then A and $\overline{B}$ are also independent.

(d) $P(A \mid B) + P(\overline{A} \mid B) = 1$ for any events A, B with $P(B) > 0$.