Chapter 12

Introduction to Probability

Intended Learning Outcomes

- Define random phenomena, outcomes, and sample spaces
- Identify and describe events
- Apply the three axioms of probability
- Use the addition rule (disjoint and general)
- Apply the complement rule
- Use Venn diagrams to visualize probabilities
- Distinguish finite and continuous probability models
- Define and interpret random variables

PART 1

From Tables to Probability

Example 12.1: From Chapter 6 to Chapter 12

Recall the flu vaccine study (Example 6.7):

Group	**Developed Flu**	**No Flu**	**Total**
Vaccinated	26	3874	3900
Control	70	3830	3900
Total	96	7704	7800

In Chapter 6, you computed:

- Marginal proportion of flu: $\frac{96}{7800} = 0.0123$
- Conditional proportion of flu given vaccinated: $\frac{26}{3900} = 0.0067$

In Chapter 12 notation:

- $P(\text{Flu}) = 0.0123$ *(probability of flu in the whole study)*
- $P(\text{Flu} \mid \text{Vaccinated}) = 0.0067$ *(probability of flu given vaccinated)*

Why Do We Need Formal Rules?

With contingency tables, we could answer questions like:

- What proportion have the flu?
- What proportion are vaccinated *and* have the flu?
- What proportion are vaccinated *or* have the flu?

But what about more complex situations?

- What if we can't enumerate all outcomes in a table?
- What if outcomes happen in stages (first this, then that)?
- What if we need to combine information from multiple sources?

Key Point: Probability theory gives us **general rules** that work in any situation, not just contingency tables.

PART 2

Foundations of Probability

Random Phenomena and Outcomes

Random Phenomenon

A **random phenomenon** is any process whose result is uncertain before it occurs.

An **outcome** is a single possible result of a random phenomenon.

Outcome not yet determined:

- Flipping a coin
- Rolling a die
- Drawing a card

Outcome determined but unknown to us:

- A patient's disease status before testing
- Tomorrow's weather
- Whether a randomly selected person is left-handed

Chapter 6 Connection
When you randomly selected a person from a contingency table, you were observing a random phenomenon. The uncertainty about their group membership is what made proportions meaningful.

Sample Space

Sample Space

The **sample space** is the set of **all possible outcomes** of a random phenomenon. We denote it by S.

Notation

We write sets using curly braces: $\{\ \}$. The braces hold all possible outcomes without regard to order or duplicates.

Quick examples:

- Flip a coin: $S = \{H, T\}$
- Roll a die: $S = \{1, 2, 3, 4, 5, 6\}$
- Pick a person from the flu study: $S = \{\text{all 7800 participants}\}$

Example 12.2: Writing Sample Spaces

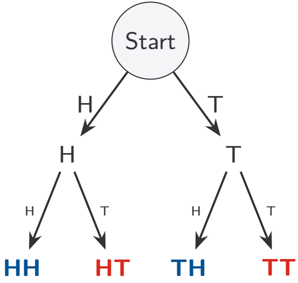
Task: Write the sample space for each random phenomenon.

1. Flipping a coin
2. Rolling a six-sided die
3. Flipping a coin twice

Tree Diagrams: Flipping Two Coins

Context: Flip a coin twice. What are all possible outcomes?

Key Point: A **tree diagram** helps visualize all possible outcomes when a random process has multiple steps.



Sample space: $S = \{HH, HT, TH, TT\}$: 4 possible outcomes.

Events

Event

An **event** is a set of outcomes from a sample space.

We rarely care about every outcome individually. Instead, we ask about **events**:

- "At least one head" (when flipping coins)
- "Rolling an even number" (when rolling a die)
- "The person has the flu" (when sampling from the flu study)

Chapter 6 Connection
In contingency tables, each row or column defined an event. "Vaccinated" was an event; "Developed Flu" was an event.

Example 12.3: Writing a Sample Space

Context: Flip a coin three times. Each flip is independent.

1. List all possible outcomes (sequences of flips):
2. Let W = "At least two heads." List the outcomes in W:

Example 12.4: Identifying Events

Task: In every experiment, list the outcomes belonging to every event.

1. Flip a coin. Event $A =$ "Get heads"
2. Roll a die. Event $B =$ "Roll an even number"
3. Draw a card. Event $C =$ "Draw a face card"
4. Flip two coins. Event $D =$ "At least one tail"

Combining Events: “Or” and “And”

Union and Intersection

- **Union** “A or B”: All outcomes in A, in B, or in both.
- **Intersection** “A and B”: Only outcomes in *both* A and B.

⚠ **Caution:** In mathematics, “or” is **inclusive**: it means at least one occurs, possibly both. This differs from everyday English where “or” often means “one or the other, but not both.”

Notation:

- **Union:** $A \cup B$ or “A or B”
- **Intersection:** $A \cap B$ or “A and B”

Example: Union and Intersection with Cards

Context: Draw one card from a standard 52-card deck.

- Event A = "Draw a heart": 13 outcomes
- Event B = "Draw a face card (J, Q, K)": 12 outcomes

Hearts (13): 2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

Face cards (12): J♣ Q♣ K♣ J♢ Q♢ K♢ J♠ Q♠ K♠ J♡ Q♡ K♡

Calculate the number of outcomes in each event:

- $A \cap B$ ("heart **and** face card"):
- $A \cup B$ ("heart **or** face card"):

Example 12.5: Combining Events

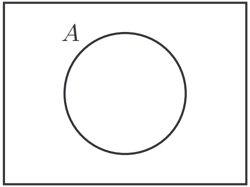
Context: Pavlov has two dogs. Each dog is male (M) or female (F), all combinations equally likely.

1. Sample space S
2. A_1: "First dog is male"
3. A_2: "Second dog is male"
4. "At least one male"
5. "Both male"

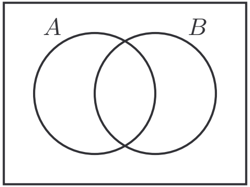
Venn Diagrams: A Visual Tool

Venn Diagram

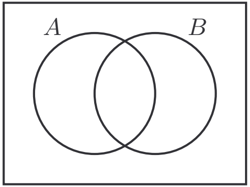
A **Venn diagram** represents events as regions inside a rectangle (the sample space). Overlapping regions show intersections.



Event A



A and B

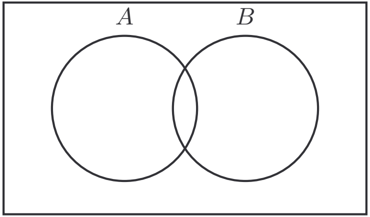


A or B

Example 12.12: Working with Venn Diagrams

Context: In a class of 30 students, 12 like apples (A), 7 like bananas (B), and 3 like both.

1. How many like only apples?
2. How many like only bananas?
3. How many like neither?



PART 3

Rules of Probability

What Is Probability?

Probability

The **probability** of an event is a number between 0 and 1 that measures how likely the event is to occur.

Three paradigms of probability:

1. **Classical:** Count equally likely outcomes.
   Example: Fair die: each face has probability $\frac{1}{6}$.
2. **Empirical:** Use long-run relative frequencies.
   Example: Track 1000 coin flips; proportion of heads ≈ 0.5.
3. **Subjective:** Based on personal judgment.
   Example: "I think there's a 70% chance it rains tomorrow."

Chapter 6 Connection

When you computed proportions from contingency tables, you were using the **empirical** approach: relative frequencies from observed data.

Example 12.6: Classical Probability

Key Point: When outcomes are equally likely:

$$P(\text{event}) = \frac{\text{number of outcomes in event}}{\text{total number of outcomes}}$$

Calculate:

- $P(\text{rolling a 4})$
- $P(\text{rolling an even number})$
- $P(\text{rolling greater than 4})$

Example 12.7: When Outcomes Are Not Equally Likely

Context: A novelty die is weighted so that 6 appears twice as often as any other number. All other faces are equally likely.

1. Find $P(6)$:
2. Find $P(\text{even number})$:
3. Find $P(\text{not a 6})$:

Which paradigm do we use?

Which paradigm do we use in practice?

- **Empirical:** When we have data (contingency tables, surveys, experiments), we use observed proportions.
- **Classical:** When outcomes are equally likely (dice, cards, lotteries), we count outcomes.

In our course, we will only use the empirical and classical paradigms.

> **Convention**
> When no information suggests otherwise, assume equally likely outcomes.

The Three Axioms of Probability

Axioms of Probability

A1: For any event A: $0 \leq P(A) \leq 1$.

A2: The probability of the entire sample space is 1: $P(S) = 1$.

A3: If A and B are **mutually exclusive** (cannot both occur), then

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 12.8: Applying the Axioms

Context: A lottery draws a three-digit number from 000 to 999. Each is equally likely.

- Total outcomes:
- Probability of each outcome:

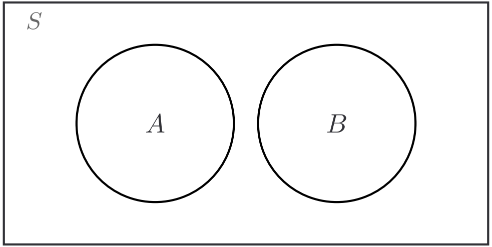
Find:

1. $P(\text{all three digits equal})$
2. $P(\text{number} < 500)$
3. $P(\text{number ends in 7})$

Mutually Exclusive Events

Mutually Exclusive (Disjoint)

Two events are **mutually exclusive** if they cannot both occur. They share no outcomes.



Example: Rolling a die: "Roll a 2" and "Roll a 5" are mutually exclusive.

Example 12.9: Checking Mutual Exclusivity

Context: Roll a fair six-sided die. Determine if each pair is mutually exclusive.

1. $A =$ "Roll a 2", $B =$ "Roll a 5"

2. $A =$ "Roll an even number", $B =$ "Roll greater than 3"

Addition Rule for Mutually Exclusive Events

Key Point: If A and B are mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B)$$

1. Roll a die. Find $P(1 \text{ or } 6)$.
2. Draw a card. Find $P(\text{heart or spade})$.

Addition Rule

Theorem: *Addition rule for multiple mutually exclusive events*

If $A_1, A_2, \ldots, A_n$ *are all mutually exclusive, then*

$$P(A_1 \text{ or } \cdots \text{ or } A_n) = P(A_1) + \cdots + P(A_n)$$

Example 12.10: General Addition Rule Example

Context: A class has 100 students. 60 students own a laptop, 40 students own a tablet, and 20 students own both.

	Tablet	**No Tablet**	**Total**
Laptop	20	40	60
No Laptop	20	20	40
Total	40	60	100

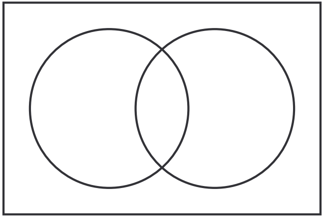
Find: $P(\text{Laptop or Tablet})$

The General Addition Rule

Addition Rule (General)

For **any** two events A and B:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



Key Point: We subtract $P(A \text{ and } B)$ because when A and B overlap, adding $P(A) + P(B)$ counts the overlap **twice**.

Example 12.13: Movie Nights

Context: At a movie night with 300 attendees:

- 120 watched a comedy (A)
- 150 watched a drama (B)
- 60 watched both a comedy and a drama $(A \cap B)$

Find: What is the probability that a randomly selected attendee watched **a comedy or a drama**?

The Complement of an Event

Complement

The **complement** of event A, written $\overline{A}$ or A^c, is the event that A does **not** occur.

Find the complement for each event:

1. $A =$ "roll a 2" on a die.
2. $B =$ "get at least one head in 3 coin flips".
3. $C =$ "draw a heart from a deck".

The Complement Rule

Complement Rule

For any event A,

$$P(\overline{A}) = 1 - P(A)$$

Example: Tossing a coin 3 times. Event A = "At least one head in 3 flips"

Example 12.11: Using the Complement Rule

Context: Roll a fair six-sided die.

1. $A =$ rolling a 2. Find $P(\overline{A})$:
2. $B =$ rolling less than 4. Find $P(\overline{B})$:

Example 12.11: Using the Complement Rule

Context: Roll a fair six-sided die.

1. Let C = "roll an even number" and D = "roll greater than 3". Find $P(C \cup D)$ and $P(\overline{C \cup D})$:

PART 4

Probability Models & Random Variables

Two Types of Probability Models

Finite Probability Model

A model whose sample space has a **finite number** of outcomes. We can list every outcome and its probability.

Continuous Probability Model

A model whose sample space is **continuous**: outcomes can take any value in an interval. We describe probabilities using a density curve.

Quick Test

Ask: "Can I list every possible outcome?"

- **Yes** → finite (e.g., number of children, die roll)
- **No** → continuous (e.g., height, time, temperature)

Finite or Continuous?

Task: Classify each scenario.

- Height of a person:
- Number of cars sold:
- Time to run a mile:
- Number of siblings:
- Goals in a hockey game:
- Body temperature:
- Eggs in a carton:
- Gas remaining in tank:

Example 12.14: A Finite Probability Model

Context: "How many cups of coffee have you had today?"

Cups (X)	0	1	2	3	4	5+
Probability	0.40	0.22	0.15	0.10	0.07	0.06

1. Verify this is a valid probability model and find
2. $P(X < 4)$ = "fewer than 4 cups"
3. $P(X \geq 1)$ = "at least one cup"

Exercise: Completing a Probability Model

Context: A discrete random variable Y has the following distribution. One probability is missing.

Value (Y)	10	20	30	40
Probability	0.15	0.30	**?**	0.10

Find the probability that $Y = 30$.

Random Variables

Random Variable

A **random variable** assigns a **numerical value** to each outcome of a random phenomenon. The **probability distribution** describes which values are possible and how likely each is.

Examples:

- X = number shown when rolling a die
- Y = number of heads in 10 coin flips
- Z = height (in cm) of a randomly selected student

⚠ **Caution:** Random variables must be **numerical**. "Favourite colour" is not a random variable (it's categorical). But we could define $X = 1$ if blue, $X = 0$ otherwise.

Example 12.15: A Discrete Random Variable

Context: Let X = number of countries visited by a randomly selected DS1000 student.

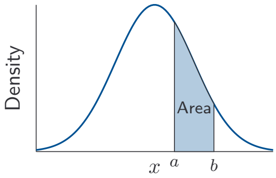
Countries	0	1	2	3	4	5	$\cdots$
Count	3	6	18	33	25	24	$\cdots$
Proportion	0.015	0.029	0.088	0.162	0.123	0.118	$\cdots$

1. $P(X = 3)$:
2. $P(X \leq 2)$:
3. $P(X > 2)$:

Continuous Random Variables

Some facts about continuous random variables:

- Continuous variables take values in an interval (e.g., height, time).
- Probabilities are represented by **area under a density curve**.
- The probability of any single exact outcome is 0.
- We calculate probabilities for intervals: $P(a < X < b)$.



$P(a < X < b)$

We'll come back to continuous random variables after Chapter 13.

CHAPTER 12

Summary

Chapter 12 Summary

Foundations

- **Random phenomenon:** uncertain outcome
- **Sample space S:** all possible outcomes
- **Event:** a set of outcomes
- **Union (or):** at least one occurs
- **Intersection (and):** both occur
- **Complement:** event does not occur

Models

- **Finite:** list outcomes in a table
- **Continuous:** use density curves
- **Random variable:** numerical outcome of a random phenomenon

Rules

- **A1:** $0 \leq P(A) \leq 1$
- **A2:** $P(S) = 1$
- **A3 (disjoint):** $P(A \bigcup B) = P(A) + P(B)$
- **General addition:** $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$
- **Complement:** $P(\overline{A}) = 1 - P(A)$

Looking Back and Looking Ahead

Key Point: Chapter 6 gave you **intuition** for probability through contingency tables. Chapter 12 gives you **formal rules** that work in any situation.

You Already Know:

- Proportions from tables
- Conditional proportions $P(A \mid B)$
- Why we subtract overlap
- Complement ("not A")

What's New:

- Formal axioms
- Sample space notation
- Tree diagrams
- Random variable notation

Coming Next

Chapter 13 will focus on **conditional probability** in depth, the formal version of the conditional proportions you computed in Chapter 6.

CHAPTER 12

Practice Problems

Example 12.17: Sample Spaces & Events

Context: A bag contains 3 marbles: Red (R), Green (G), and Blue (B). You draw two marbles one at a time **without replacement**.

Find:

1. List the sample space S. (Hint: use a tree diagram.)
2. Let $A =$ "Red is drawn at some point." List the outcomes in A.
3. Let $B =$ "Green is drawn first." List the outcomes in B.
4. Find $P(A)$, $P(B)$, and $P(A \cap B)$.
5. Are A and B mutually exclusive? Justify.

Example 12.18: Venn Diagrams

Context: A survey of 200 university students found:

- 90 use Instagram (I)
- 110 use TikTok (T)
- 50 use both

Find:

1. Fill in all four regions of a Venn diagram with counts.
2. $P(I \cup T)$ (uses Instagram or TikTok)
3. $P(\overline{I \cup T})$ (uses neither)
4. P(uses exactly one of the two platforms)

Example 12.19: Finite Probability Models

Context: Let X = number of pets owned by a randomly selected student. The probability distribution is:

Pets (X)	0	1	2	3	4+
Probability	0.18	0.35	0.25	**?**	0.07

Find:

1. Find $P(X = 3)$.
2. Find $P(X \geq 2)$.
3. Find $P(X < 2)$.
4. Is X a finite or continuous random variable? Justify.

Example 12.20: Weighted Outcomes

Context: A spinner has four sections: Red, Blue, Green, and Yellow. Red is twice as likely as Blue. Green and Yellow are each equally likely as Blue.

Find:

1. Find the probability of landing on each colour.
2. $P(\text{Red or Green})$. Are these events mutually exclusive?
3. $P(\text{not Yellow})$.

Example 12.21: Languages

Context: At a company, $P(\text{speaks French}) = 0.30$, $P(\text{speaks Spanish}) = 0.25$, and $P(\text{speaks French or Spanish}) = 0.45$.

Find:

1. $P(\text{speaks both French and Spanish})$
2. $P(\text{speaks neither French nor Spanish})$
3. Are "speaks French" and "speaks Spanish" mutually exclusive? Justify.

Example 12.22: Another password

Context: A password consists of one letter (A, B, or C) followed by one digit (1, 2, 3, or 4). All passwords are equally likely.

Find:

1. How many outcomes are in the sample space?
2. P(password starts with A)
3. P(password contains an even digit)
4. P(starts with A **or** contains an even digit)
5. P(does not start with A **and** contains an odd digit)

Example 12.23: More events

Context: For events A and B: $P(A) = 0.6$, $P(B) = 0.5$, and $P(\overline{A \cup B}) = 0.1$.

Find:

1. $P(A \cup B)$
2. $P(A \cap B)$
3. $P(\text{exactly one of } A \text{ or } B)$
4. Are A and B mutually exclusive? Justify.