DS 1000B: Chapter 12 Review

Introduction to Probability

PART 1

The Language of Probability

Key vocabulary and concepts

Random Phenomena and Sample Spaces

Random Phenomenon

A process whose result is uncertain before it occurs. Each possible result is called an **outcome**.

Notation: ω or s.

Sample Space S

The set of **all possible outcomes** of a random phenomenon.

Notation: S or Ω.

The sample space must be **exhaustive** (include all outcomes) and **mutually exclusive** (no outcome can occur more than once).

Quick example:

- Flip a coin twice: $S = \{HH, HT, TH, TT\}$
- Measure someone's height: $S = \{x : x > 0\}$

Events

Event

An **event** is a set of outcomes from a sample space.

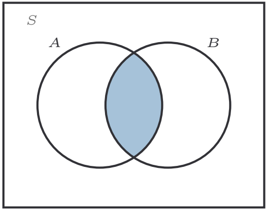
We use capital letters (A, B, $\ldots$) to name events.

We rarely care about individual outcomes. Instead, we ask about events:

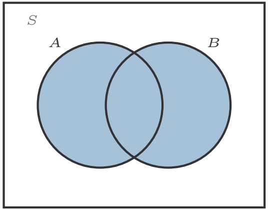
- "At least one head" (when flipping coins): $\{HH, HT, TH\}$
- "Roll an even number" (when rolling a die): $\{2, 4, 6\}$

Combining Events

Symbol	Read as	Meaning
$A \cap B$	"A and B"	Both A and B occur
$A \cup B$	"A or B"	At least one of A, B occurs



$A \cap B$: **AND**



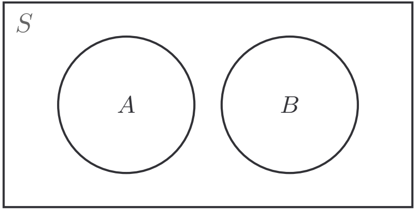
$A \cup B$: **OR**

Caution: "Or" in mathematics is **inclusive**: at least one, possibly both.

Mutually Exclusive Events

Mutually Exclusive (Disjoint)

Two events are **mutually exclusive** if they share no outcomes: $A \cap B = \emptyset$.



Mutually exclusive

Quick example: Roll a die.

- $A =$ "roll a 2", $B =$ "roll a 5"

  Mutually exclusive

- $A =$ "even", $B =$ "greater than 3"

  Not mutually exclusive

PART 2

The Rules of Probability

The three axioms and their consequences

The Three Axioms of Probability

All of probability is built on three rules.

Axiom 1. For any event A: $\quad 0 \leq P(A) \leq 1.$

Axiom 2. $P(S) = 1.$

Axiom 3. If A and B are mutually exclusive:

$$P(A \cup B) = P(A) + P(B)$$

What Do the Axioms Mean?

The axioms seem simple, but they are powerful.

- **Axiom 1** says probabilities live on a scale from 0 (impossible) to 1 (certain).

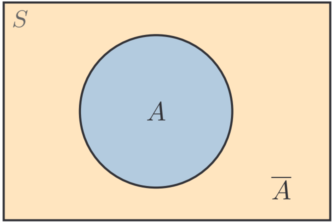
  No event can have a probability of -0.3 or 1.5.
- **Axiom 2** says that an outcome in the sample space must occur.
- **Axiom 3** says for non-overlapping events, the probability of "one or the other" is the sum of the individual probabilities. This extends to any number of disjoint events:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

The Complement Rule

Complement

The **complement** of event A, written $\overline{A}$, is the event that A does **not** occur.



Since A and $\overline{A}$ are disjoint and together make up all of S, the axioms give:

$$P(\overline{A}) = 1 - P(A)$$

Key Point: Use the complement when it is easier to calculate what you do **not** want.

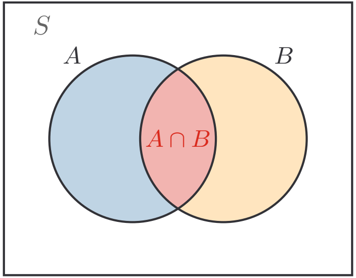
The General Addition Rule

Axiom 3 only works when events are disjoint. What if they overlap?

General Addition Rule

For **any** two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



We subtract $P(A \cap B)$ because adding $P(A) + P(B)$ counts the overlap **twice**.

When A and B are disjoint, $P(A \cap B) = 0$, so this reduces to Axiom 3.

Example: Movie Nights

Context: At a movie night with 300 attendees:

- 120 watched a comedy
- 150 watched a drama
- 60 watched both

Find: probability of a randomly selected attendee watching a comedy or a drama.

Example: Laptops and Tablets

Context: A class has 100 students. 60 own a laptop, 40 own a tablet, and 20 own both.

	Tablet	No Tablet	Total
Laptop	20	40	60
No Laptop	20	20	40
Total	40	60	100

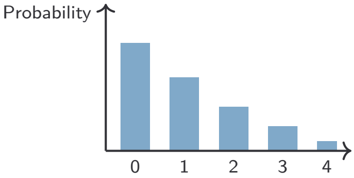
Find: probability of a randomly selected student having a laptop or a tablet.

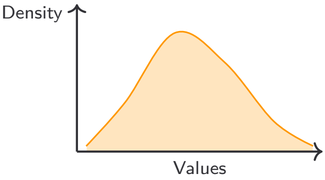
PART 3

From Rules to Random Variables

Why probability distributions must sum to 1

Finite Probability Models





Finite Probability Model

- A countable number of outcomes.
- Each outcome is assigned a probability.
- All probabilities must sum to 1.
- Ask: can you list every possible value?

Continuous Probability Model

- Values span an interval (e.g., height, time).
- Probabilities are **areas** under a density curve.
- Total area under the curve equals 1.
- Ask: does the variable take values on a continuous scale?

Discrete Probability Models (Preview)

Discrete Random Variable

A finite probability model is part of a larger family of probability models known as **discrete** probability models.

In these models, the outcomes take on a value which is **countable**, meaning it is either finite or can be put into a one-to-one correspondence with the natural numbers (countably infinite list like $0, 1, 2, 3, \ldots$).

Common examples you may encounter in future courses include the **binomial** and **Poisson** distributions.

What Is a Random Variable?

Random Variable

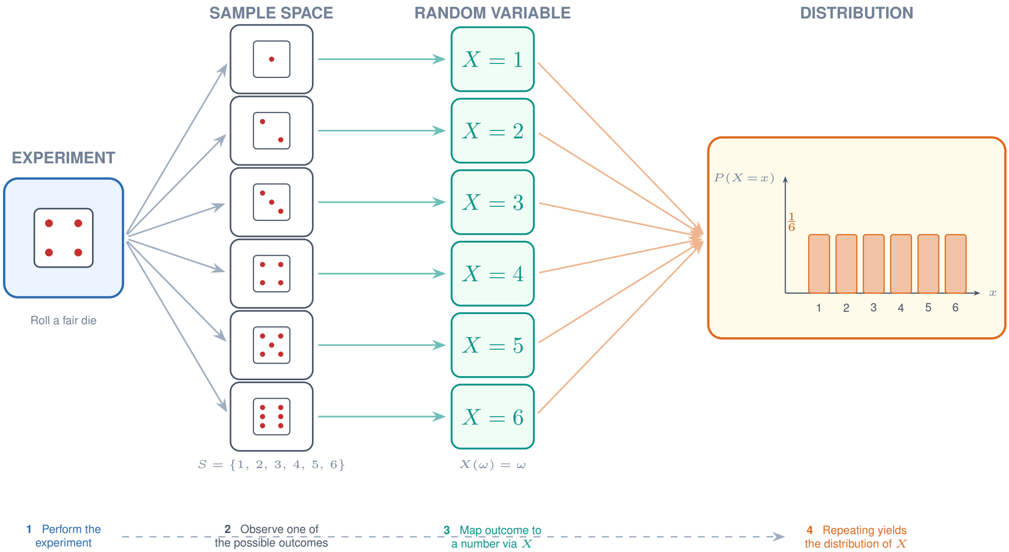
A **random variable** assigns a **numerical value** to each outcome of a random phenomenon. We use capital letters: X, Y, Z.

Quick example: Flip a coin twice. Let $X =$ number of heads.

Outcome	Value of X	Event
TT	0	$\{X = 0\}$
HT	1	$\{X = 1\}$
TH	1	
HH	2	$\{X = 2\}$

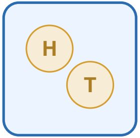
Each value of X defines an **event**: a set of outcomes from the sample space. This is the key idea we will build on next.

Rolling a die



Tossing two coins

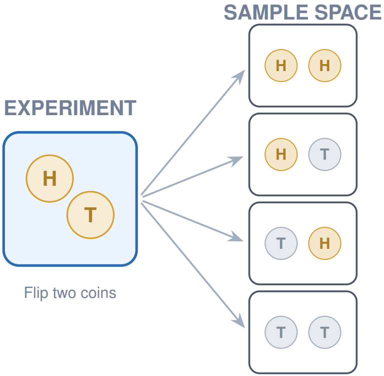
EXPERIMENT



Flip two coins

1 Perform the experiment

Tossing two coins

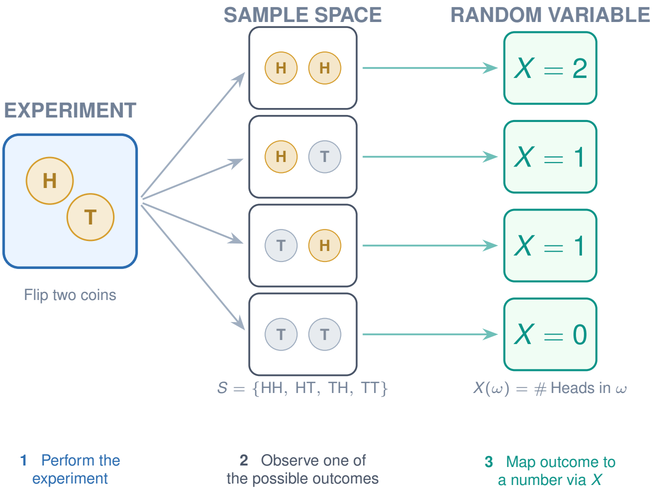


$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

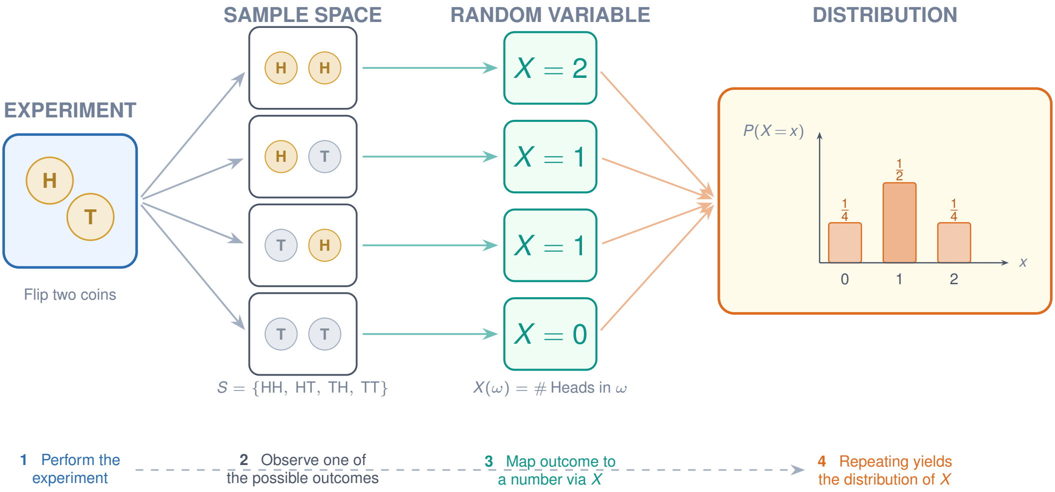
1 Perform the experiment

2 Observe one of the possible outcomes

Tossing two coins

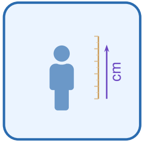


Tossing two coins



Measuring height

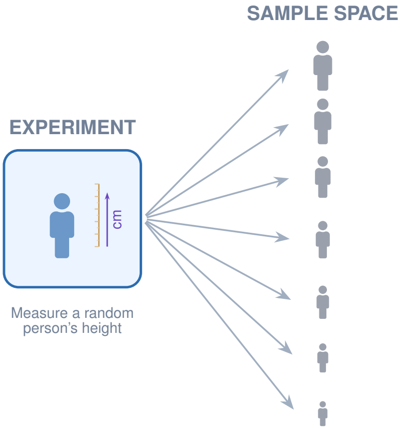
EXPERIMENT



Measure a random person's height

1 Perform the experiment

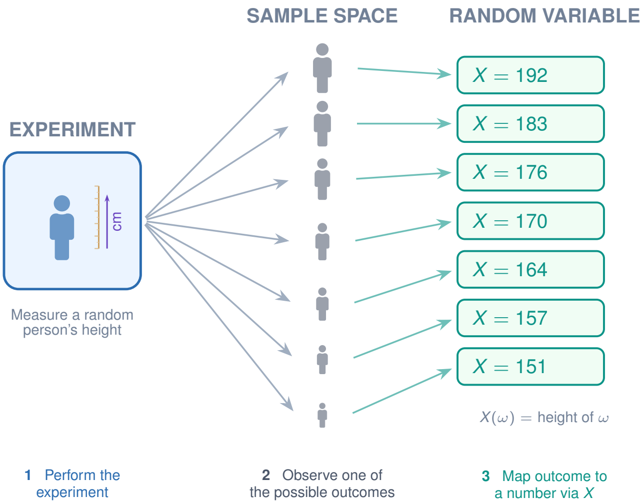
Measuring height



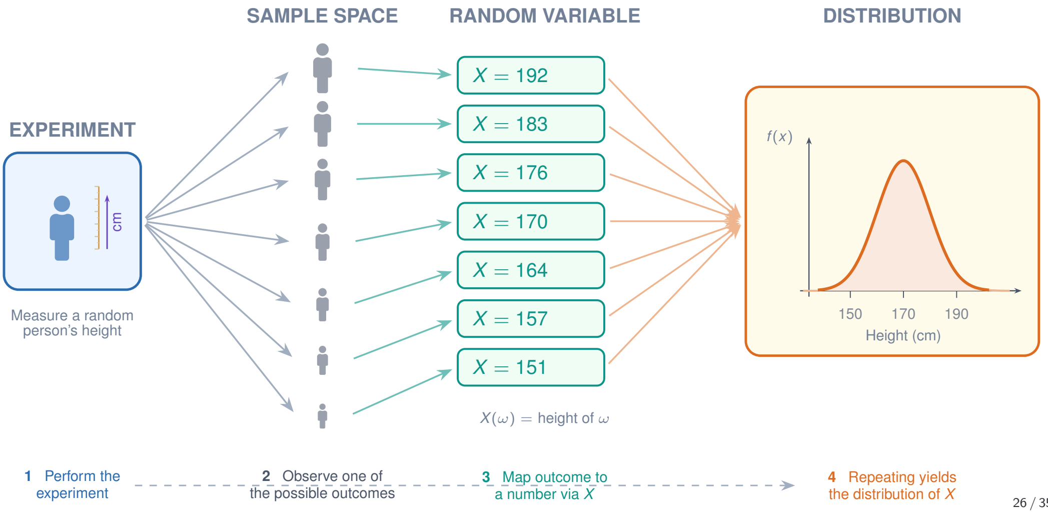
1 Perform the experiment

2 Observe one of the possible outcomes

Measuring height



Measuring height



Building a Probability Distribution

Context: Flip a fair coin twice. $X =$ number of heads. All 4 outcomes are equally likely, each with probability $\frac{1}{4}$.

The **(probability) distribution** of X lists every possible value and its probability:

Value of X	0	1	2
Outcomes in event	TT	HT, TH	HH
$P(X)$	$1/4$	$2/4$	$1/4$

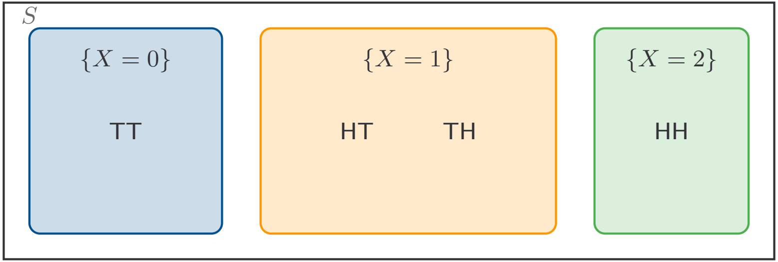
Notice that the probabilities sum to 1:

$$\frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1.$$

Is this true in general?

Why Must Probabilities Sum to 1? Setup

Let's think carefully about what the events $\{X = 0\}$, $\{X = 1\}$, and $\{X = 2\}$ look like inside the sample space.



Two observations about these events:

1. They are **mutually exclusive**: no outcome belongs to two groups at once.
   (You cannot get 0 heads and 1 head simultaneously.)
2. They are **exhaustive**: every outcome in S belongs to exactly one group.
   (Every flip sequence produces *some* number of heads.)

In other words: $\{X = 0\} \cup \{X = 1\} \cup \{X = 2\} = S$.

Why Must Probabilities Sum to 1?

The General Pattern

The same logic works for **any** finite random variable.

If X takes possible values $x_1, x_2, \ldots, x_k$, then:

- The events $\{X = x_1\}, \{X = x_2\}, \ldots, \{X = x_k\}$ are **mutually exclusive** (a random phenomenon produces exactly one value of X).
- Their union is S (every outcome maps to some value).

Therefore, by Axioms 2 and 3:

$$P(X = x_1) + P(X = x_2) + \cdots + P(X = x_k) = 1$$

The Distribution of a Finite Random Variable

If X is a finite random variable with probability distribution given by:

Value	x_1	x_2	$\cdots$	x_k
Probability	p_1	p_2	$\cdots$	p_k

Then

- Each p_i satisfies $0 \leq p_i \leq 1$ (Axiom 1).
- The probabilities sum to 1 (Axioms 2 and 3):

$$p_1 + p_2 + \cdots + p_k = 1$$

Example: Using the Rules with Random Variables

Context: X takes values 0, 1, and 2 with $P(X = 0) = 0.3$, $P(X = 1) = 0.5$, and $P(X = 2)$ unknown.

Find the following

(1) the missing probability $P(X = 2)$

(2) $P(1 \leq X \leq 2)$

Worked Example: Coffee Survey

Context: X = number of cups of coffee a student had today.

Cups (X)	0	1	2	3	4	5
Probability	0.40	0.22	0.15	0.10	0.07	0.06

(a) What needs to be true for this to be a valid probability distribution?

(b) Find $P(X \geq 1)$ = "at least one cup."

(c) Find $P(X < 4)$ = "fewer than 4 cups."

CHAPTER 12

Summary

Key ideas from this review

Chapter 12 Summary

Language

- **Sample space S:** all possible outcomes
- **Event:** a set of outcomes
- **Union ($\cup$):** at least one occurs
- **Intersection ($\cap$):** both occur
- **Mutually exclusive:** $A \cap B = \emptyset$

Rules

- **Axiom 1:** $0 \leq P(A) \leq 1$
- **Axiom 2:** $P(S) = 1$
- **Axiom 3:** Disjoint $\Rightarrow$ add probabilities
- **General addition:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Complement:** $P(\overline{A}) = 1 - P(A)$

Random Variables

- Assigns a numerical summary to each outcome (e.g., number of heads or height)
- The events $\{X = x_i\}$ are mutually exclusive and exhaustive
- The distribution of a finite random variable must satisfy: