Chapter 3

The Normal Distribution

Intended Learning Outcomes

- Understand what a density curve is and its properties
- Describe a Normal distribution using $N(\mu, \sigma^2)$
- Apply the 68–95–99.7 rule
- Compute and interpret z-scores
- Use the standard Normal table to find proportions
- Find the kth percentile of a Normal distribution (inverse Normal)

PART 1

Density Curves

What is a smooth mathematical model for data, and how do we describe it?

Continuous Distributions: Motivation

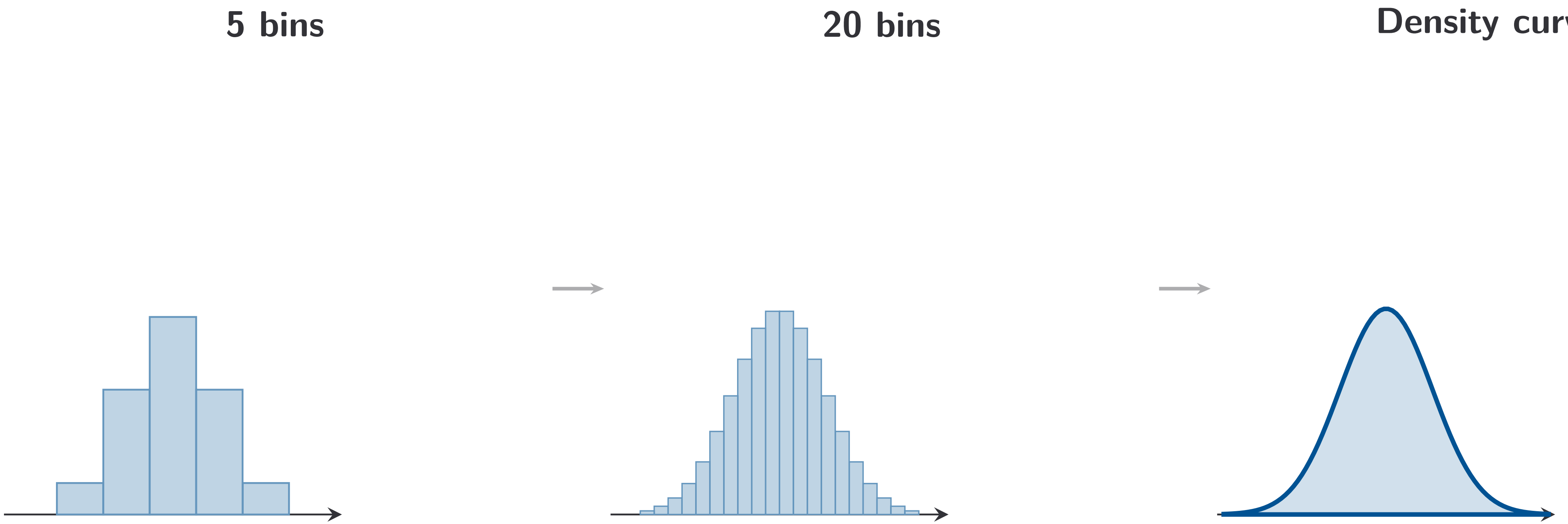
Most real-world measurements (like screen time, testosterone levels, highway speeds) can take any value within a range, not just whole numbers. These are called **continuous distributions**.

- Discrete: countable outcomes (e.g., number of downloads of an app)
- Continuous: any value in an interval (e.g., daily screen time in hours)

Density Curves

Example 3.1

Density curves are descriptions of the probability distribution of a continuous random variable. As the number of bins increases, the histogram approaches a smooth curve:



A density curve is the limiting shape of a histogram as the sample size grows and the bins narrow. It models the overall pattern of data without depending on specific observed values.

Density Curves: Definition

Density Curve

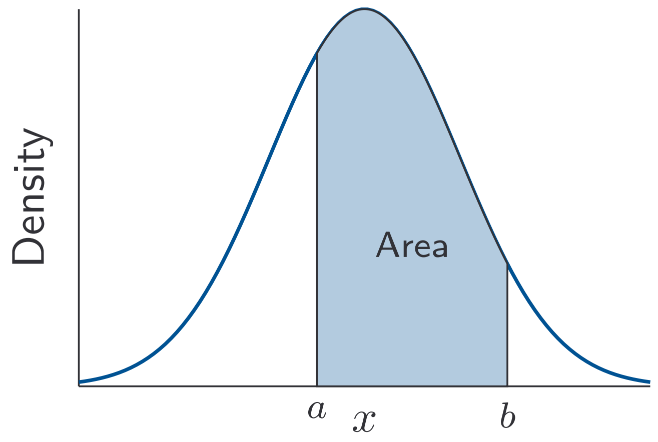
A **density curve** is a curve that:

- is always on or above the horizontal axis (i.e. non-negative)
- has a total area of exactly **1** underneath it

(Probability is nonnegative)

(2nd axiom of Probability)

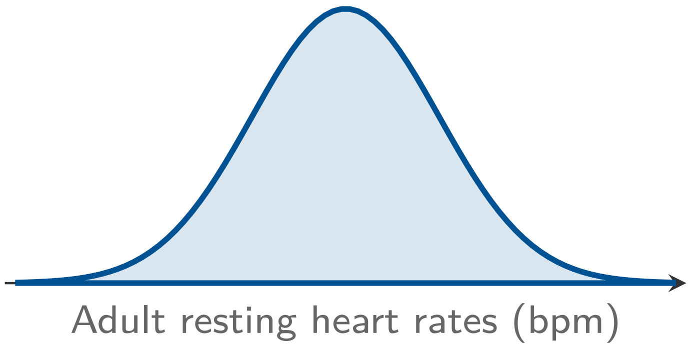
- A density curve is a mathematical model for the distribution of a continuous random variable.
- The **area** under the curve above any range of values equals the **proportion** (or probability) of observations in that range.



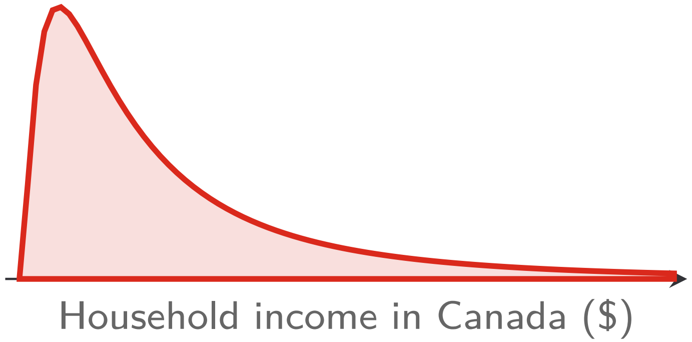
Area between a and b = proportion of observations falling in $[a, b]$.

A variety of density curves exist in the world

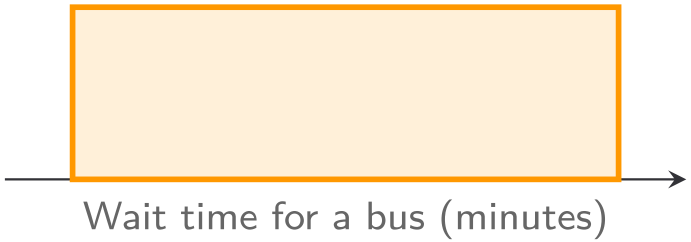
Bell-shaped



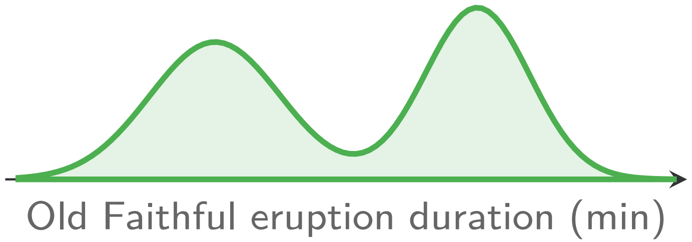
Right-skewed



Uniform



Bimodal



A Simple Density Curve

Example 3.2

Omitted

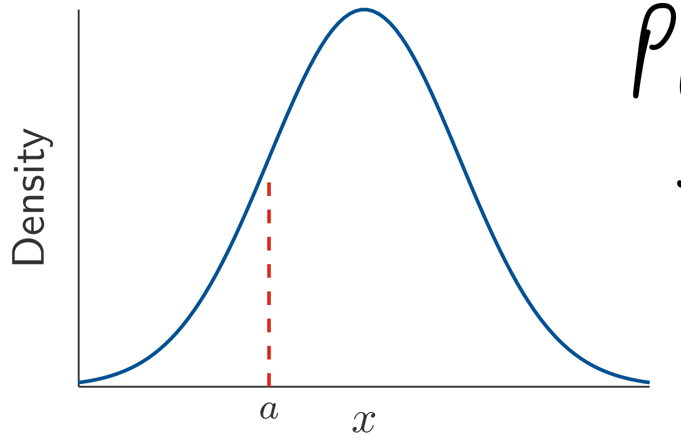
Context: A city bus arrives every 10 minutes. A passenger arrives at a random time and waits up to 10 minutes for the next bus. The wait time X is uniformly distributed on $[0, 10]$, so the density curve has height $\frac{1}{10} = 0.1$ everywhere on that interval.

(a) How do we verify this is a valid density curve?

(b) Find $P(X \leq 6)$: the probability of waiting less than 6 minutes.

Important note regarding continuous distributions

> **Principle:** For continuous distributions, the probability of observing any single exact value is zero.



$P(X = a)$
$= \text{Area under } f(x) \text{ between } a \text{ and } a$
$= f(a) \cdot 0 = 0$

The probability of X taking any specific value a is the area under the curve at exactly a, which is zero because it's just a line with no width.

We only have non-zero probabilities for intervals of values, which correspond to areas under the curve.

For any continuous r.v. X, $P(X=a) = 0$.

Probabilities of Intervals for continuous distributions

As a consequence of the previous principle, it follows that the following expressions are equivalent for any values a and b:

- $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
- $P(X \leq b) = P(X < b)$
- $P(X \geq a) = P(X > a)$

From Data to Mathematical Models

In chapter 2, we have described data using **statistics** computed from a sample:

- sample mean $\bar{x}$
- sample standard deviation s
- the five-number summary

But to understand the bigger picture, we need to describe the **entire population** or the **theoretical model** (like a density curve) that generates the data. To do this, we need a different kind of number.

A sample yields STATISTICS:

- A mean $\bar{x}$ that varies between samples
- A standard deviation s that varies between samples
- Numbers that describe *the data we collected*

A population/model has PARAMETERS:

- A fixed center μ (the true mean)
- A fixed spread σ (the true SD)
- Numbers that describe *the entire population or model*

PART 2

The Normal Distribution

The bell curve: parameters, shape, and the 68–95–99.7 rule

Normal Distributions

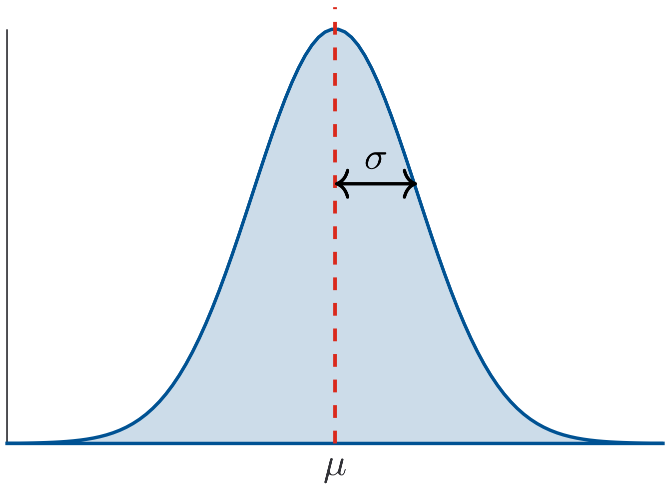
Normal Distribution

A **Normal distribution** is described by a symmetric, bell-shaped density curve. It is completely determined by its mean μ and standard deviation σ:

$$X \sim N(\mu, \sigma^2)$$

↑ variance

Aside: $N(\mu, \sigma)$
$G(\mu, \sigma)$



Normal distributions describe many real measurements: screen time, testosterone levels, caffeine content in coffee.

Notation for Normal Distributions

The Standard Notation:

$$X \sim N(\mu, \sigma^2)$$

- **Read as:** "X follows a Normal distribution with mean μ and variance σ^2."
- **The tilde ($\sim$):** Means "*is distributed according to*."
- Note: the second parameter is the variance σ^2, not the standard deviation σ. There are two equivalent ways to express the same distribution (e.g. $N(100, 15^2)$ and $N(100, 225)$ are the same distribution).

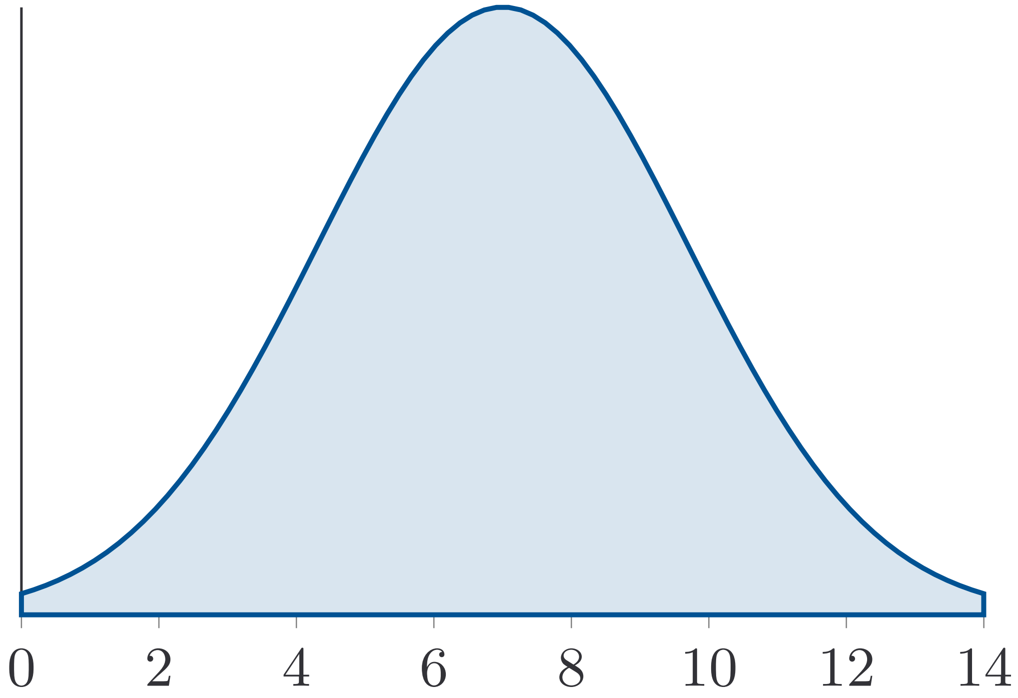
Theoretical Parameters vs. Sample Estimates:

- μ and σ are fixed parameters that define the theoretical model.
- In practice, these are often estimated from data using the sample mean ($\bar{x}$) and sample standard deviation (s).

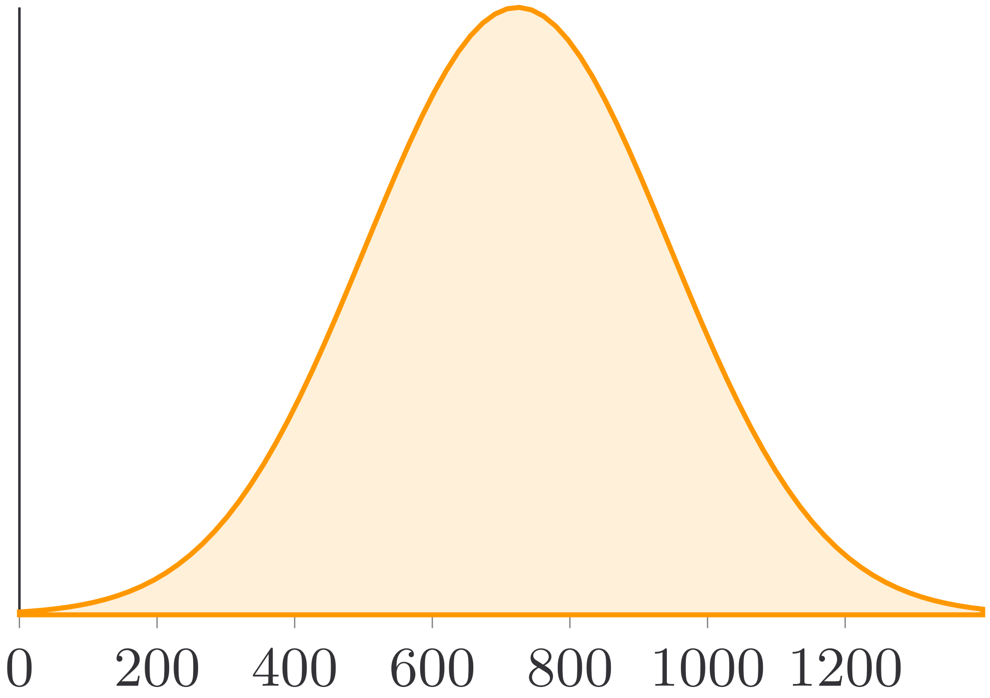
Why Is the Normal Distribution So Common?

Principle: When many small, independent factors contribute to a measurement, the result tends to follow a Normal distribution.

Screen Time (hours/day)



Testosterone (ng/dL)

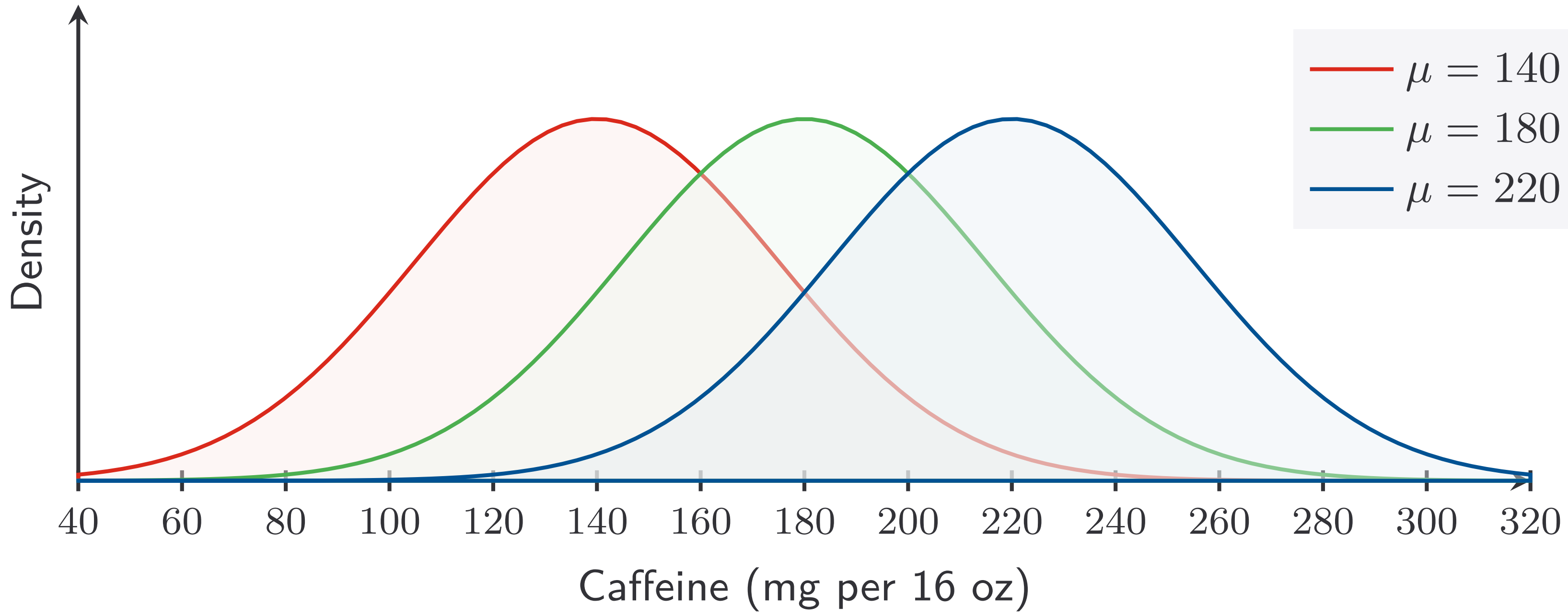


The Effect of μ

Example 3.3

Context: A food scientist measures the caffeine content (in mg per 16 oz cup) at three coffee chains. All three chains have similar cup-to-cup variability ($\sigma = 35$ mg), but different average caffeine levels.

Task: Look at the three curves below. What changes when μ changes? What stays the same?

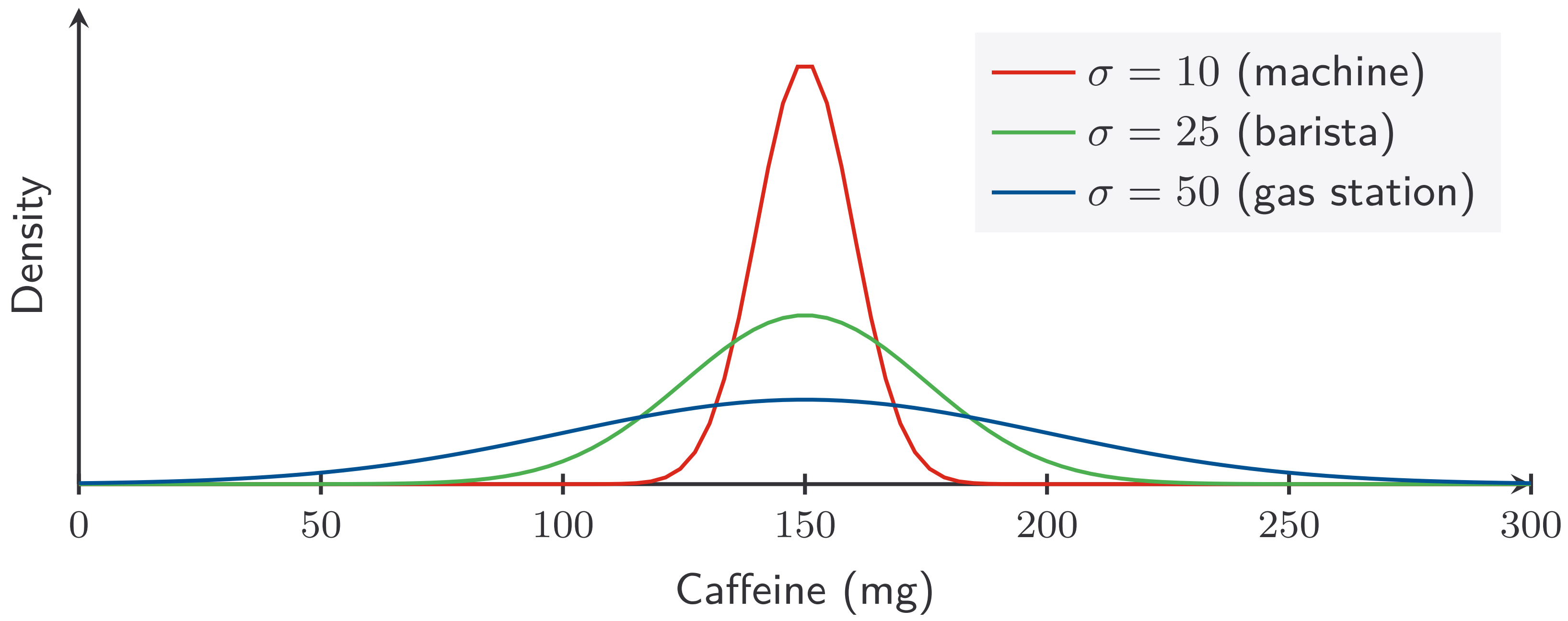


The Effect of σ

Example 3.4

Context: Three different espresso drinks all have the same average caffeine content ($\mu = 150$ mg), but vary in consistency.

Task: Look at the three curves. What changes when σ changes? Which espresso drink is most consistent?

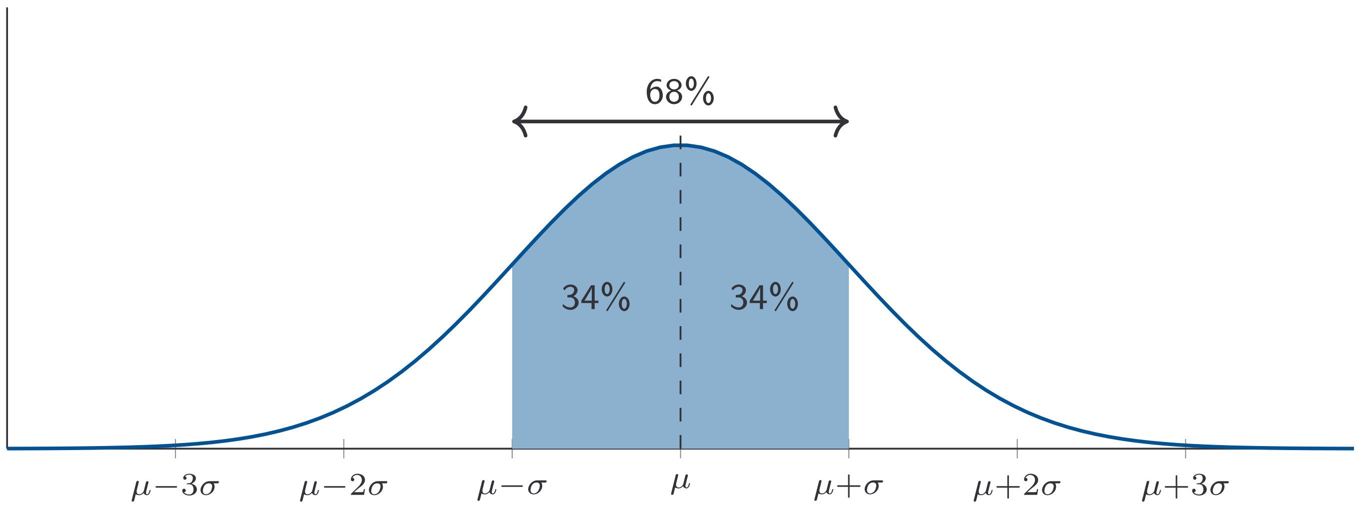


The Empirical Rule

The 68–95–99.7 Rule

In a Normal distribution $N(\mu, \sigma^2)$:

- Approximately **68%** of observations fall within $\mu \pm \sigma$

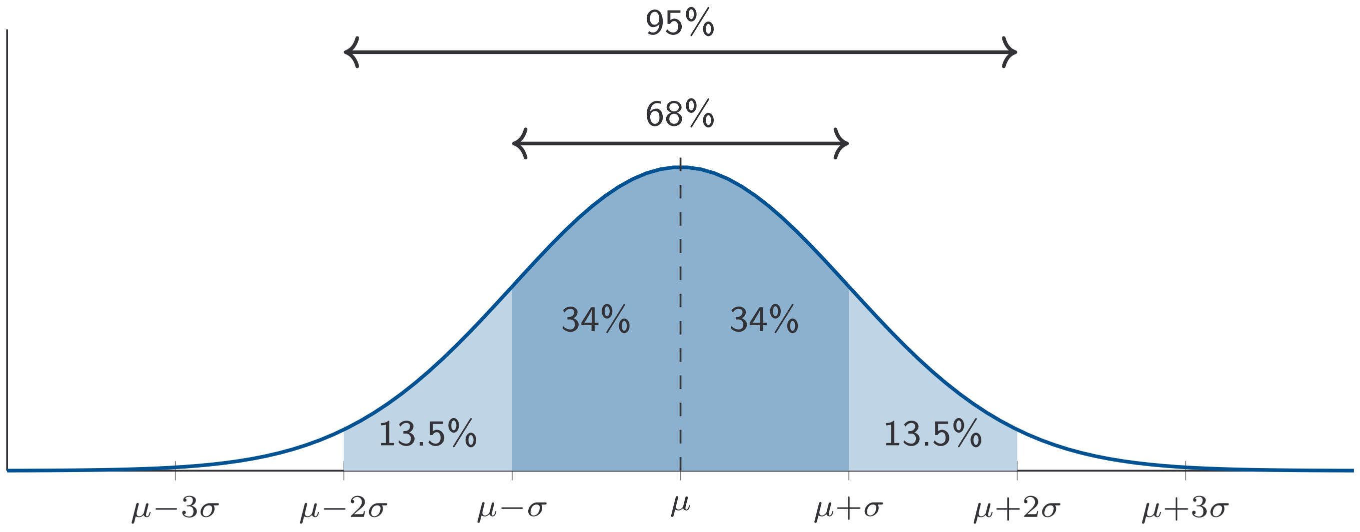


The Empirical Rule

The 68–95–99.7 Rule

In a Normal distribution $N(\mu, \sigma^2)$:

- Approximately **68%** of observations fall within $\mu \pm \sigma$
- Approximately **95%** of observations fall within $\mu \pm 2\sigma$

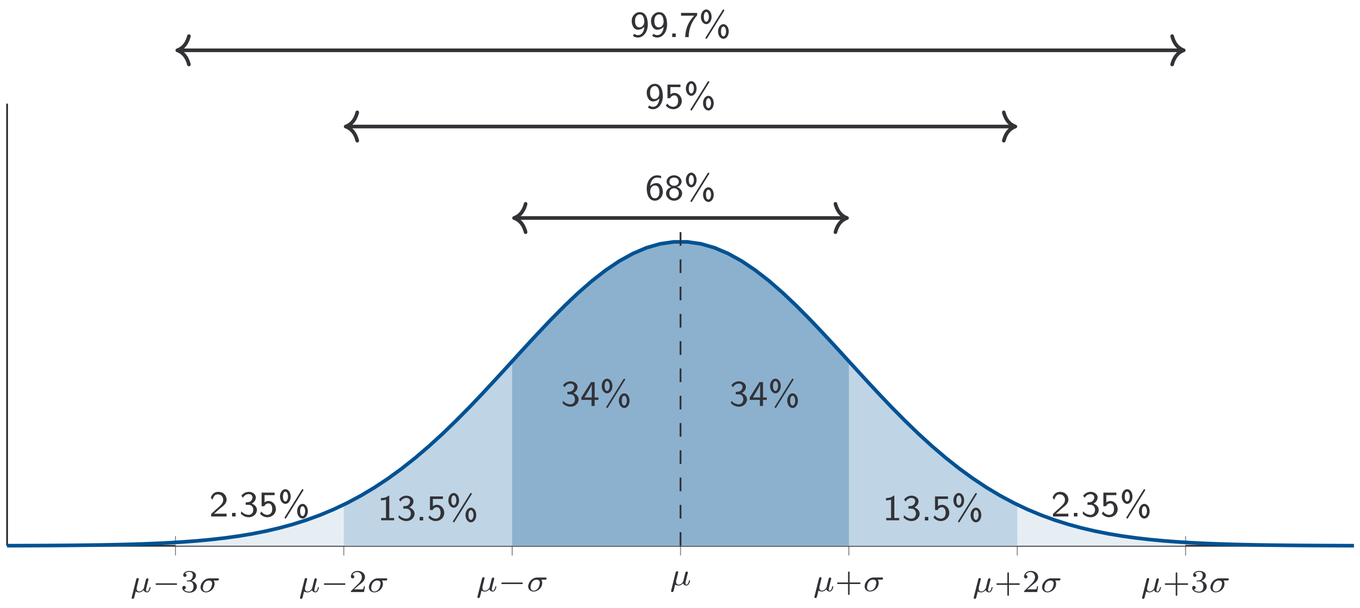


The Empirical Rule

The 68–95–99.7 Rule

In a Normal distribution $N(\mu, \sigma^2)$:

- Approximately **68%** of observations fall within $\mu \pm \sigma$
- Approximately **95%** of observations fall within $\mu \pm 2\sigma$
- Approximately **99.7%** of observations fall within $\mu \pm 3\sigma$



Practice with the 68–95–99.7 Rule

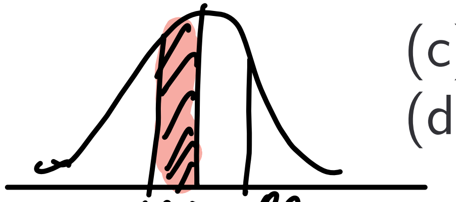
Example 3.5

$\sigma = 10$

Context: Resting heart rate among healthy adults is approximately $N(70, 100)$ bpm.

Find: What percentage of adults have a resting heart rate:

(a) Between 60 and 70 bpm?
(b) Between 70 and 90 bpm?
(c) Below 70 bpm?
(d) Above 50 bpm?



let $X \sim N(70, 100)$

a) $P(60 \le X \le 70) = P(60 \le X \le 70)$
$= \frac{1}{2} P(60 \le X \le 80) = \frac{1}{2} 0.68 = 0.34$

b) $P(70 \le X \le 90) = \frac{1}{2} P(50 \le X \le 90$
$= \frac{1}{2} 0.95 = 0.475$

Practice with the 68–95–99.7 Rule

Example 3.5

Context: Resting heart rate among healthy adults is approximately $N(70, 100)$ bpm.

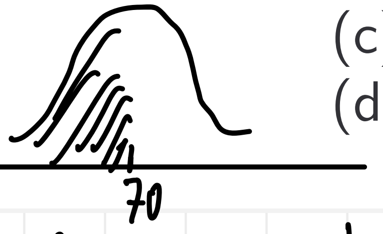
Find: What percentage of adults have a resting heart rate:

(a) Between 60 and 70 bpm?

(b) Between 70 and 90 bpm?

(c) Below 70 bpm?

(d) Above 50 bpm?



c) $P(X \le 70) = \frac{1}{2} P(X \le 70, \text{ or, } X > 70)$

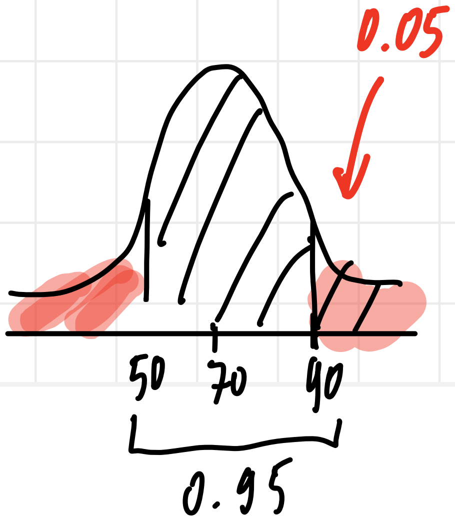
$= \frac{1}{2} \cdot 1 = \frac{1}{2}.$

d) $P(X > 50) = 1 - P(X \le 50)$

($50 = \mu - 2\sigma$)

$= 1 - 0.025$

$= 0.975$



Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

PART 3

Standardizing and z-Scores

How many standard deviations from the mean

Standardizing: From Any Normal to One Table

Principle: All Normal distributions are the same if we measure in units of σ from the mean μ. This is called **standardizing**.

The z-Score

If x is an observation from a distribution with mean μ and standard deviation σ, its **standardized value** (z-score) is:

$$z = \frac{x - \mu}{\sigma}$$

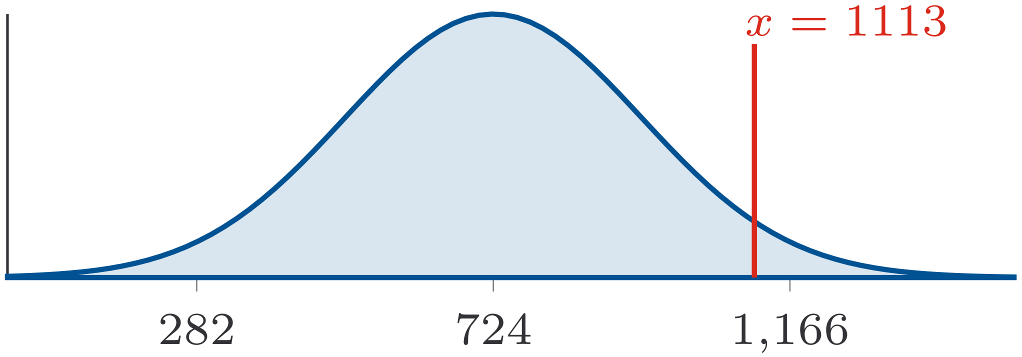
$$Z = (X - \mu)/\sigma \iff X = \mu + \sigma Z$$

- z tells you **how many standard deviations** x is from the mean
- $z > 0$: above the mean; $z < 0$: below; $z = 0$: at the mean

$$\sigma Z = X - \mu \iff \mu + \sigma Z = X$$

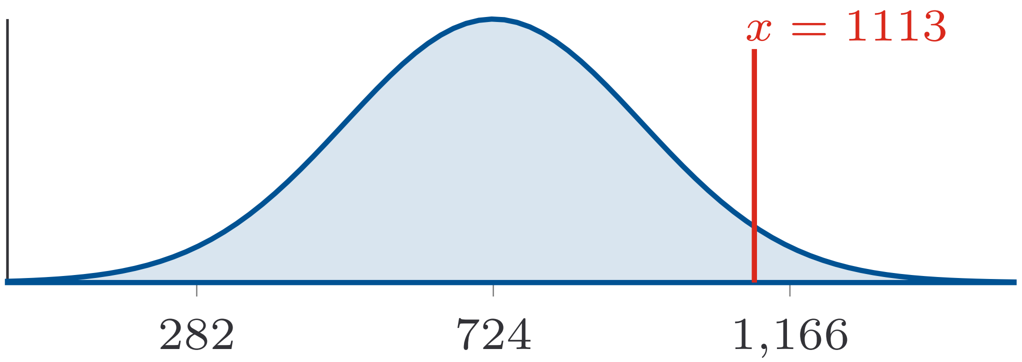
Standardizing: Visualized

Original: $N(724, 221^2)$



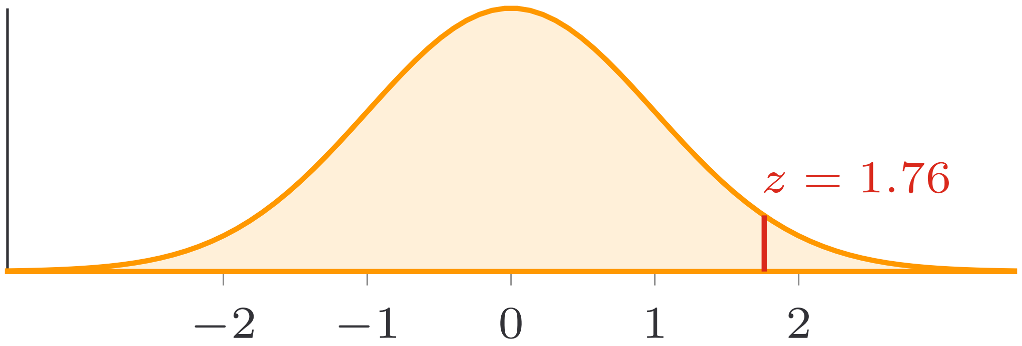
Standardizing: Visualized

Original: $N(724, 221^2)$



$z = \frac{x-\mu}{\sigma}$

Standardized: $N(0, 1)$



Computing z-Scores

Example 3.6

Context: Testosterone levels among men aged 19–29 are approximately $N(724, 221^2)$ ng/dL.

Find: the z-score for each of the following testosterone levels.
(a) A man has 950 ng/dL.

(b) A man has 400 ng/dL.

From z-Score to Observation

Example 3.7

Context: Testosterone levels among men follow $N(724, 221^2)$ ng/dL. Testosterone levels among women follow $N(33, 12^2)$ ng/dL.

Find: the actual testosterone level (in ng/dL) for each of the following.

(a) A man with $z = 1.2$:

(b) A woman with $z = -0.8$:

(c) A woman with $z = -2.1$:

(d) A man with $z = 3.0$:

Comparing z-Scores Across Groups

Example 3.8

Context: Testosterone levels among men follow $N(724, 221^2)$ ng/dL. Testosterone levels among women follow $N(33, 12^2)$ ng/dL.

Find: Whose testosterone level is more unusual relative to their group: a man at 950 ng/dL, or a woman at 55 ng/dL?

Man, 950 ng/dL:

$z =$

Woman, 55 ng/dL:

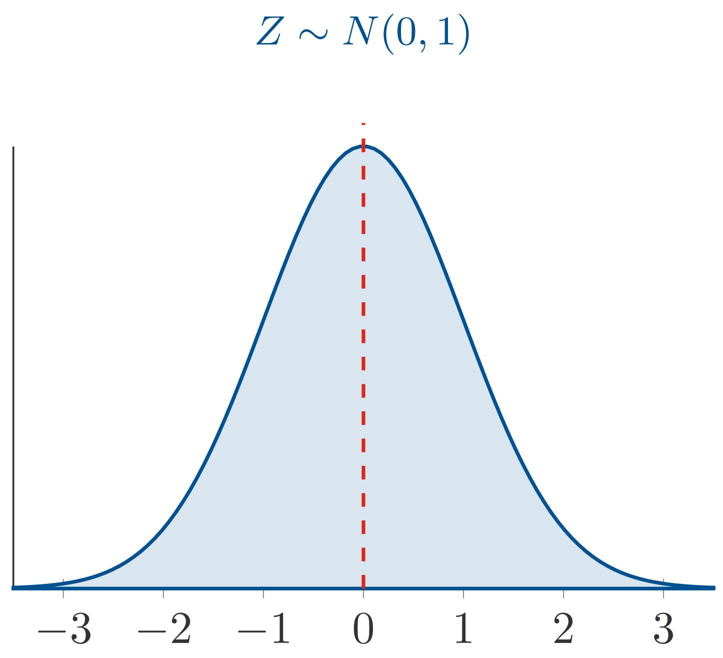
$z =$

The Standard Normal Distribution

Standard Normal Distribution

The **standard Normal distribution** is the Normal distribution with mean 0 and standard deviation 1:

$$Z \sim N(0, 1)$$



Why the Standard Normal?

Principle: Every Normal distribution can be converted to $N(0, 1)$ by standardizing.

This means we can compute any normal probability using just the distribution of the standard normal.

Without standardizing:

- Need a separate table for every (μ, σ) combination
- Testosterone: table for $N(724, 221^2)$
- Screen time: table for $N(7, 2.7^2)$
- Caffeine: table for $N(188, 36^2)$

$\vdots$

- Infinitely many tables

With standardizing:

- Convert any x to $z = \frac{x-\mu}{\sigma}$
- Look up $\Phi(z)$ in one standard Normal table
- Works for any Normal distribution
- One table handles every problem

PART 4

Calculating Probabilities of the Normal Distribution

Using standard normal table to find exact areas under the curve

The standard normal distribution

- Any $N(\mu, \sigma^2)$ can be standardized to $N(0,1)$ using $z = \frac{x-\mu}{\sigma}$
- So we only ever need one distribution to compute probabilities from normal random variables (the standard Normal distribution)

For example, if we want to calculate $P(X \leq 1113)$ where $X \sim N(724, 221^2)$, we can rewrite this in terms of Z:

- $X \sim N(724, 221^2)$.
- $P(X \leq 1113) = P(X - 724 \leq 1113 - 724)$

$$= P\left(\frac{X-724}{221} \leq \frac{1113-724}{221}\right)$$

$$= P(Z \leq 1.76)$$

$Z \to N(0,1)$

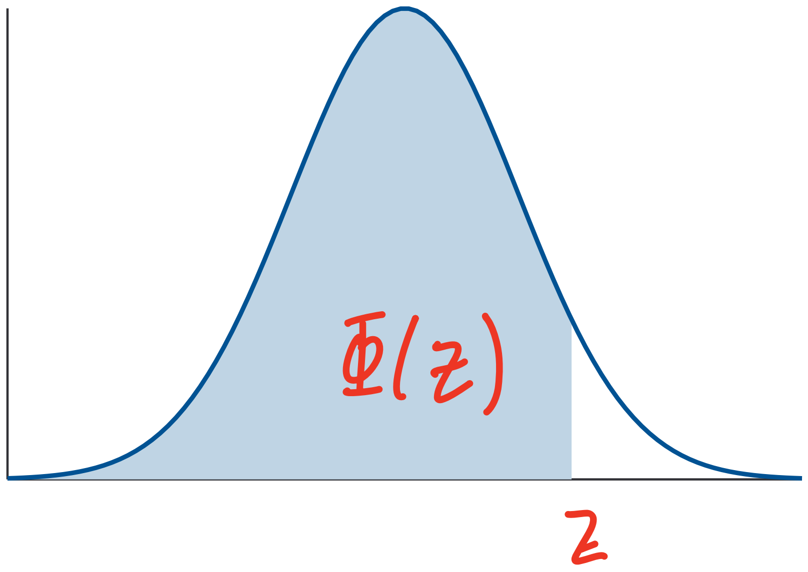
If we could evaluate $P(Z \leq 1.76)$, we could also evaluate $P(X \leq 1113)$

The Φ Notation

$\Phi(z)$: *Phi of z*

We denote $\Phi(z)$ for the probability that a standard Normal random variable is less than or equal to z:

$$\Phi(z) = P(Z \leq z) \quad \text{where } Z \sim N(0,1)$$



Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Goal: $P(Z \le 2.49)$

Decoding the standard normal table

Given a value of z, the table gives $\Phi(z) = P(Z \le z)$

$z = 2.49$

- Row: 2.4
- Column: 0.09

The table covers $z = 0.00$ to $z = 3.49$.

$$P(Z \le 2.49) = \Phi(2.49) = 0.9936$$

Reading the Standard Normal Table

Example 3.9

Find: Find the probability $P(Z \leq 1.76)$ in the standard normal table.

Find: $\Phi(1.76) = P(Z \leq 1.76)$.

Row 1.7, column .06:

$\Phi(1.76) =$ 0.9608

STANDARD NORMAL TABLE

z	$\cdots$	.04	.05	.06	.07	.08	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1.5	$\cdots$	.9382	.9394	.9406	.9418	.9429	$\cdots$
1.6	$\cdots$	.9495	.9505	.9515	.9525	.9535	$\cdots$
1.7	$\cdots$	.9591	.9599	.9608	.9616	.9625	$\cdots$
1.8	$\cdots$	.9671	.9678	.9686	.9693	.9699	$\cdots$
1.9	$\cdots$	.9738	.9744	.9750	.9756	.9761	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Reading the Standard Normal Table: More Practice

Example 3.9

Find: Use the standard normal table to find each probability.

(a) $P(Z \leq 0.94)$

① Look up row 0.9
② Check the col 0.04

$\Rightarrow \Phi(0.94) = 0.8264$

(b) $P(Z \leq 1.23)$

Exercise

Answer: $\Phi(1.23) = 0.8907$

Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
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2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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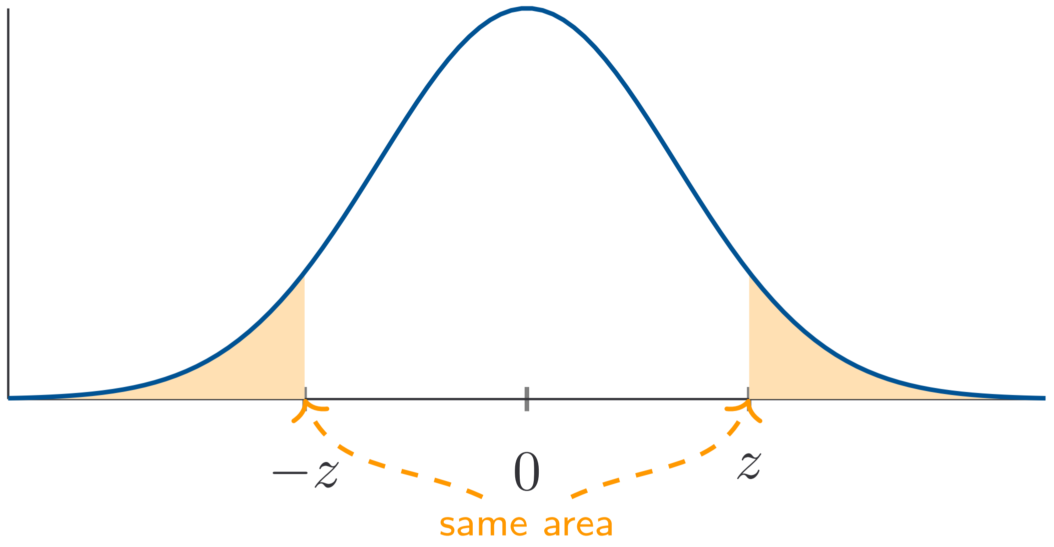
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Using the Table When z Is Negative

Our table only lists $\Phi(z)$ for $z \geq 0$. The curve's symmetry handles negative z.



Symmetry Rule

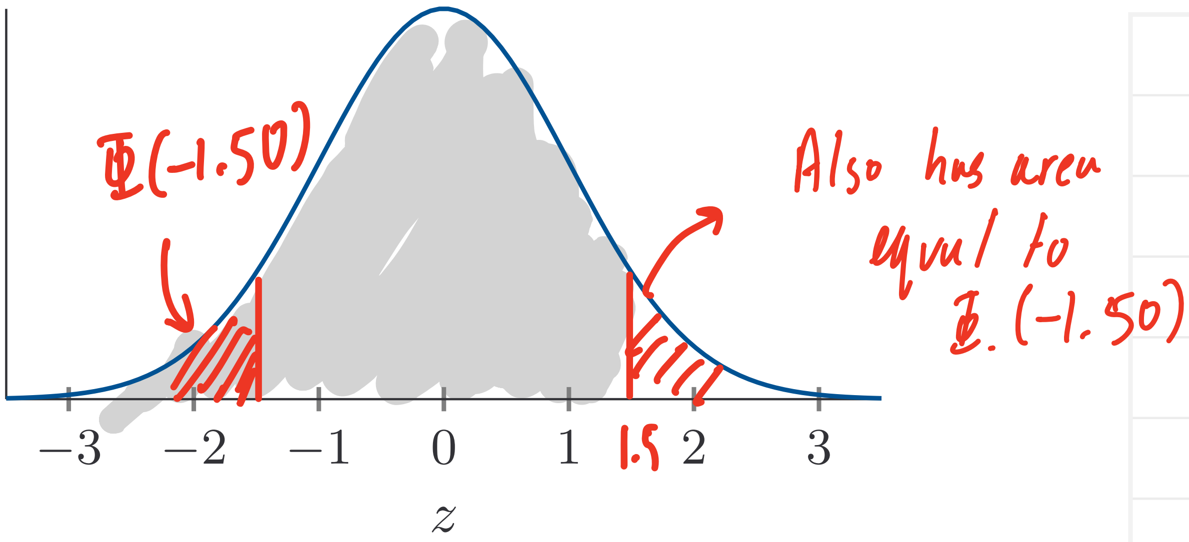
For any $z > 0$:

$\Phi(z) = 1 - \Phi(-z)$

Suggestion: visualize the desired probabilities on a density curve.

Quick Example: $\Phi(-1.50)$

Find: Find $\Phi(-1.50)$.



To find $\Phi(-1.50)$ we compute

$$1 - \Phi(1.50) = 1 - 0.9332 = 0.0668.$$

Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Standard Normal Table

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1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
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1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
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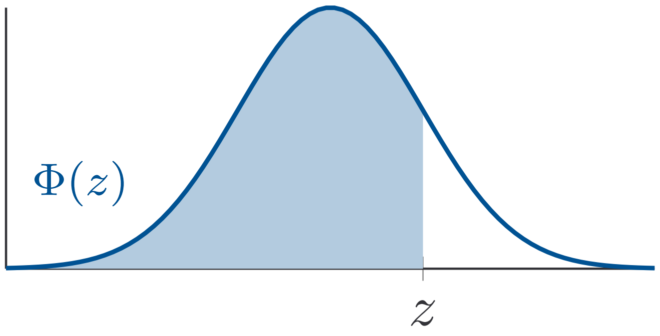
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Three Types of Normal Calculations

When working with the Normal distribution, there are three main types of probability calculations:

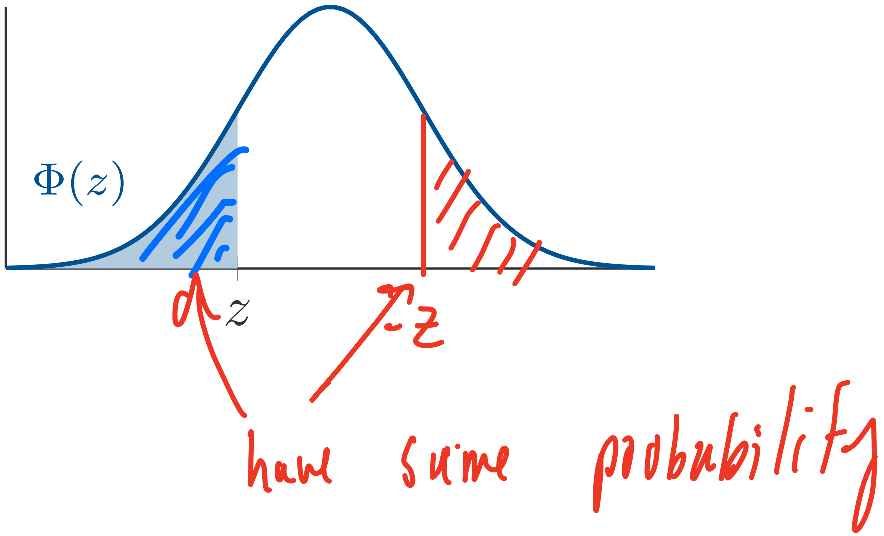
- **Left-tail:** $P(X \leq x)$: the proportion of observations less than or equal to a value.
- **Right-tail:** $P(X \geq x)$: the proportion of observations greater than or equal to a value.
- **Between:** $P(a \leq X \leq b)$: the proportion of observations between two values.

Left Tail: $P(Z \leq z)$



When $z > 0$ and $z \leq 3.49$, use the table.

(We just want the corresponding entry)



When $z < 0$, we use symmetry:

$$P(Z \leq z) = P(Z \geq -z)$$
$$= 1 - P(Z \leq -z)$$
$$= 1 - \Phi(-z)$$

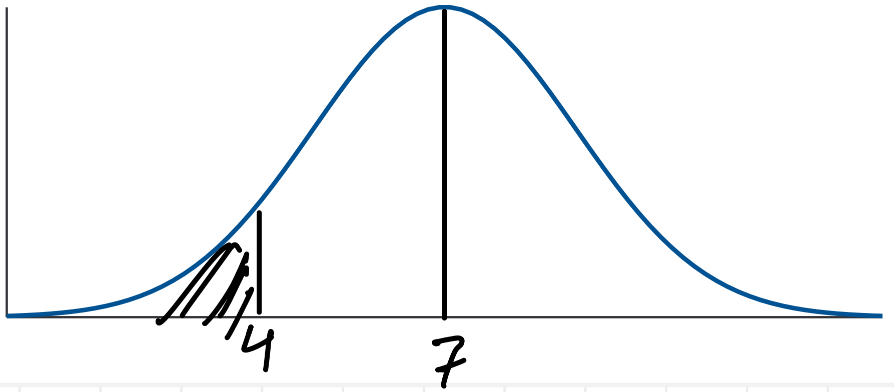
Screen Time: Under 4 Hours per Day

Context: Daily smartphone screen time among college students is approximately $N(7, 2.7^2)$ hours.
Find: What proportion of students use their phone less than 4 hours per day?

Let X be daily smartphone screen time.

$X \sim N(7, 2.7)$

Goal: Find $P(X \leq 4)$



$$= P\left(\frac{X-7}{2.7} \leq \frac{4-7}{2.7}\right)$$

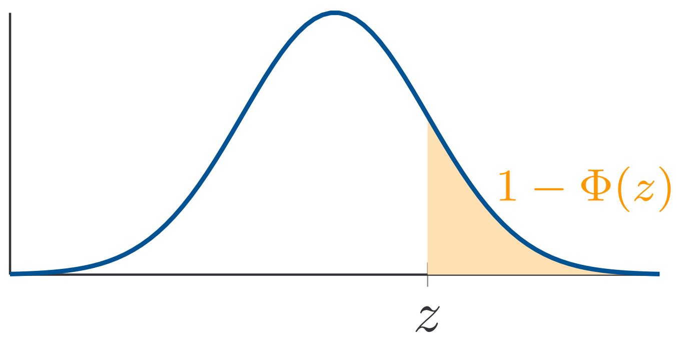
$$= P(Z \leq -1.11) = P(Z \geq 1.11)$$

$$= 1 - P(Z \leq 1.11)$$

$$= 1 - \Phi(1.11)$$

$$= 1 - 0.8665 = 0.1335$$

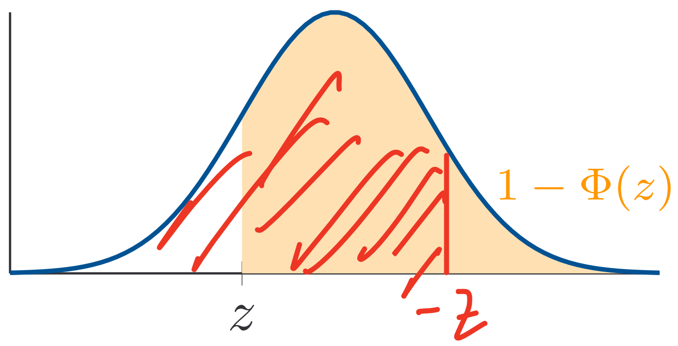
Right Tail: $P(Z \geq z)$



When $z > 0$:

$$P(Z \geq z) = 1 - \Phi(z)$$

↙ from table



When $z < 0$:

$$P(Z \geq z) = P(Z \leq -z) = \Phi(-z)$$

Screen Time: Over 12 Hours per Day

Example 3.10

Context: Daily smartphone screen time among college students is approximately $N(7, 2.7^2)$ hours.

Find: What proportion of students spend more than 12 hours per day on their phone?

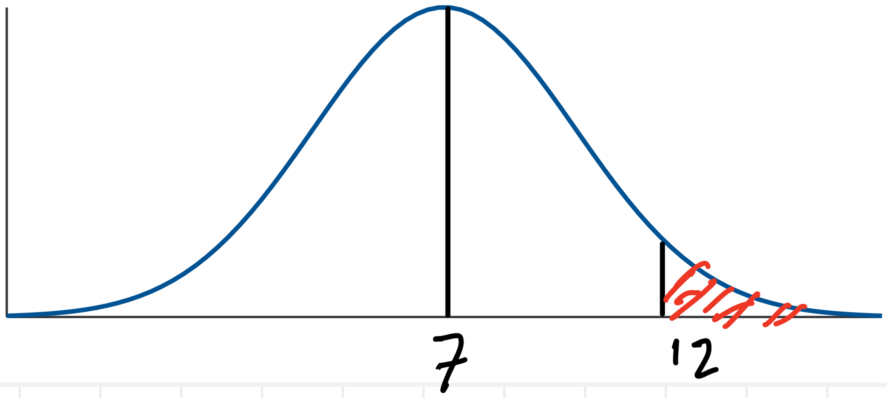
$X \sim N(7, 2.7)$

Goal: find $P(X \geq 12)$

$= P\left(\frac{X-7}{2.7} \geq \frac{12-7}{2.7}\right)$

$= P(Z \geq 1.85)$

$= 1 - P(Z \leq 1.85) = 1 - \Phi(1.85) = 1 - 0.9678 = 0.0322$



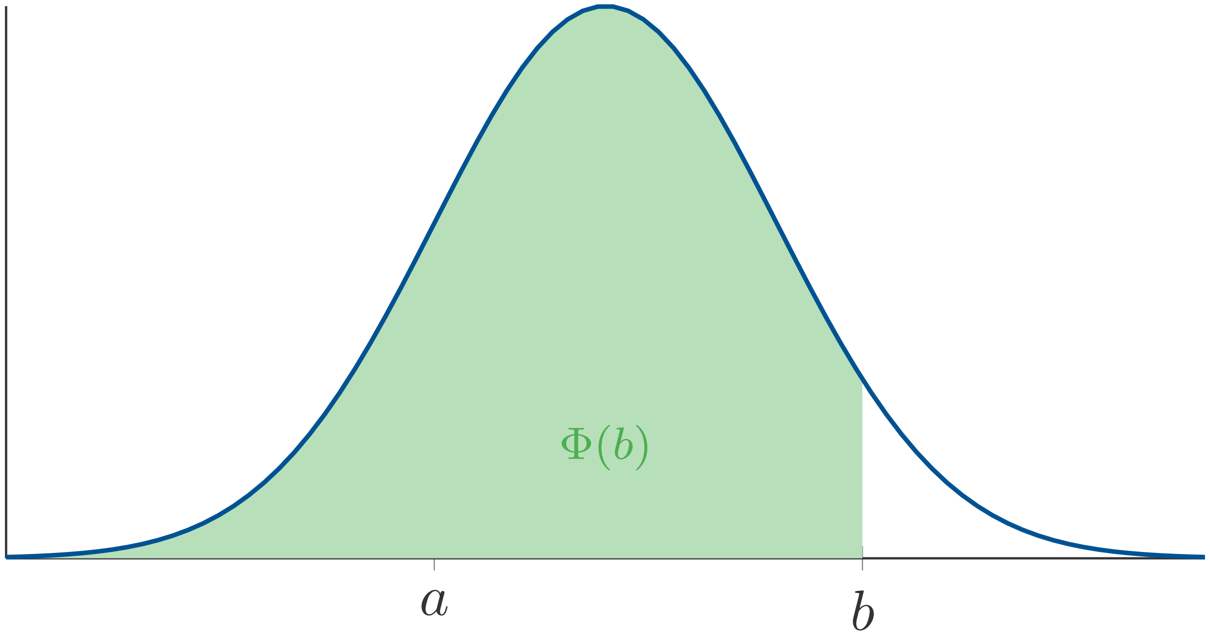
Standard Normal Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Between Two Values: $\Phi(b) - \Phi(a)$

Look up both values, then subtract.

$$P(a \leq Z \leq b) = \Phi(b) - \Phi(a)$$

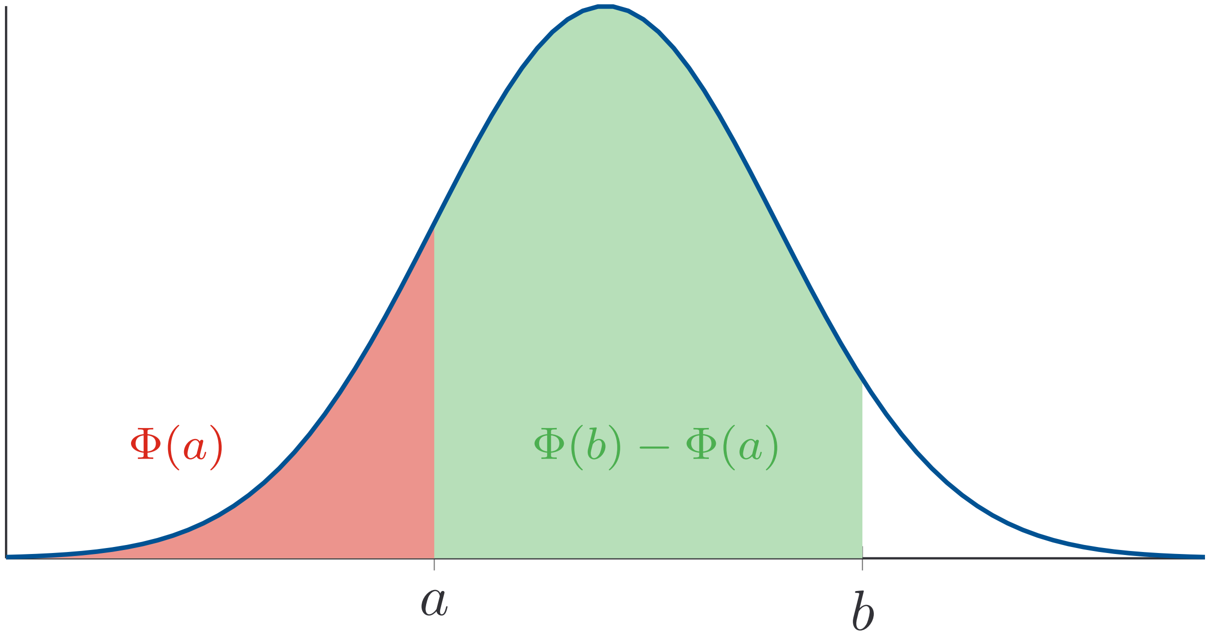


The green area represents $\Phi(b)$: the cumulative probability at b.

Between Two Values: $\Phi(b) - \Phi(a)$

Look up both values, then subtract.

$$P(a \leq Z \leq b) = \Phi(b) - \Phi(a)$$



Subtracting $\Phi(a)$ (red) leaves only the probability between a and b (green).

Screen Time: Between 5 and 10 Hours per Day

Example 3.11

Context: Daily smartphone screen time among college students is approximately $N(7, 2.7)$ hours.
Find: What proportion of students spend between 5 and 10 hours per day on their phone?

Goal: Find $P(5 \le X \le 10)$

$$= P\left(\frac{5-7}{2.7} \le \frac{X-7}{2.7} \le \frac{10-7}{2.7}\right)$$

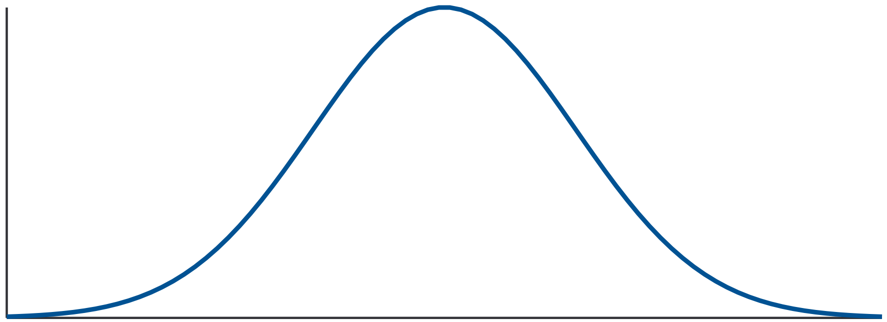
$$= P(-0.74 \le Z \le 1.11)$$

$$= P(Z \le 1.11) - P(Z \le -0.74)$$

$$= \Phi(1.11) - (1 - P(Z \le 0.74))$$

$$= \Phi(1.11) - 1 + \Phi(0.74) = \Phi(1.11) + \Phi(0.74) - 1.$$

$$= 0.6369$$



Three-Step Method: Finding a Probability of the Normal Distribution

Use this when you know a value and want to find what proportion of the distribution is above or below it.

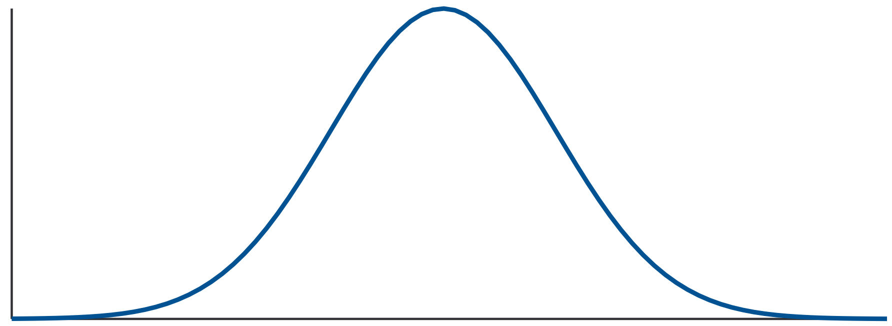
Computing a normal probability

1. **Standardize:** Convert x-values to z-scores.
2. **State and sketch:** Draw the standard Normal curve, mark the value(s) of interest, and shade the area you want.
3. **Identify the type:** Is this a left-tail, right-tail, or between calculation? If any z-score is negative, apply the symmetry rule.
4. **Look up and compute:** Use the standard Normal table to find the required area.

Caffeine in Coffee: Exceeding 250 mg

Example 3.12

Context: A food-safety researcher measured the caffeine content of 16 oz specialty coffees. Cups contain on average 188 mg of caffeine, with a standard deviation of 36 mg, following $N(188, 36^2)$ mg.
Find: What percentage of cups exceed 250 mg (a level associated with anxiety symptoms)?

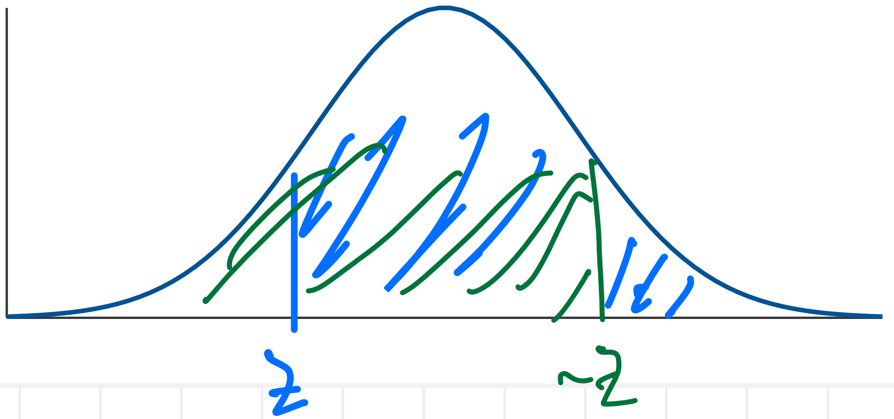


Screen Time: Under 3 Hours per Day

Example 3.13

Context: Daily smartphone screen time among college students is approximately Normally distributed with a mean of 7 hours and a standard deviation of 2.7 hours.

Find: What percentage of students use their phone less than 3 hours per day?

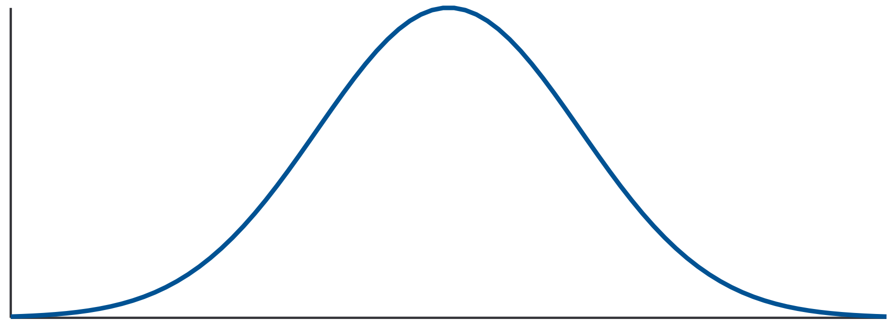


$$P(Z \geq z) = P(Z \leq -z)$$
$$= \Phi(-z)$$

Screen Time: Between 4 and 10 Hours

Example 3.14

Context: Daily smartphone screen time among college students is approximately Normally distributed with a mean of 7 hours and a standard deviation of 2.7 hours.
Find: What percentage of students spend between 4 and 10 hours per day on their phone?



PART 5

Normal Percentile Calculations

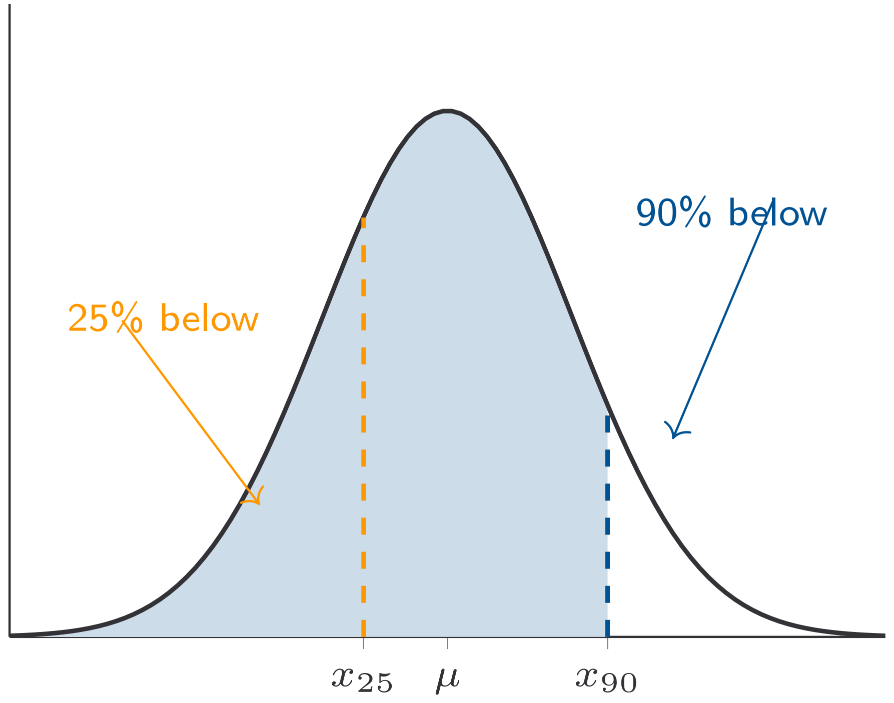
Given a proportion, what value has this much area below it?

Percentiles

Percentile

The k**th percentile** of a distribution is the value x such that $k\%$ of observations fall at or below x.

- $P(X \leq x) = k/100$
- The 50th percentile is the **median**
- The 25th and 75th percentiles are the **quartiles**
- A z-score and a percentile convey the same information in different units



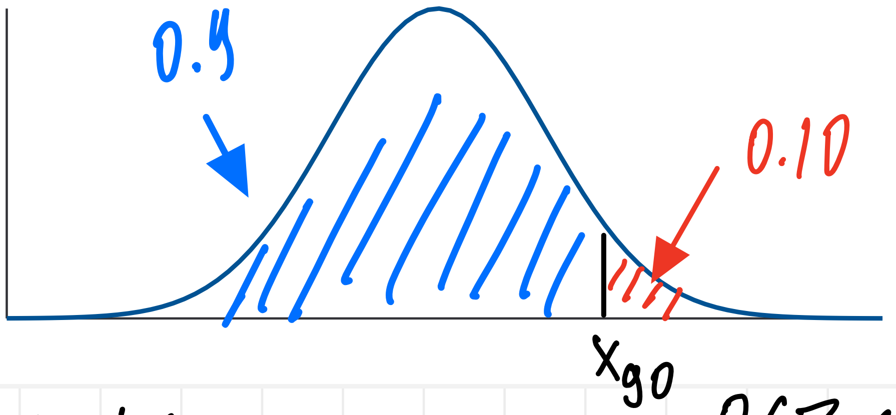
Highway Speed: Fastest 10% of Drivers

Example 3.15

$X \sim N(117, 8)$

Context: Vehicle speeds on a 100 km/h highway are approximately $N(117, 8^2)$ km/h.
Find: Above what speed are the fastest 10% of drivers?

We want x_{90} where x_{90} satisfies $P(X \leq x_{90}) = 0.90$



Note: ① $Z = \dfrac{X-\mu}{\sigma}$

$\Rightarrow \sigma Z = X - \mu$

$\Rightarrow \mu + \sigma Z = X$

② The std normal table gives us $P(Z \leq z)$

$P(Z \leq z_{90}) = 0.90$

$= P(\sigma Z \leq \sigma z_{90})$

$= P(\mu + \sigma Z \leq \mu + \sigma z_{90})$

$= P(X \leq \underbrace{\mu + \sigma z_{90}}_{x_{90}}) = 0.90$

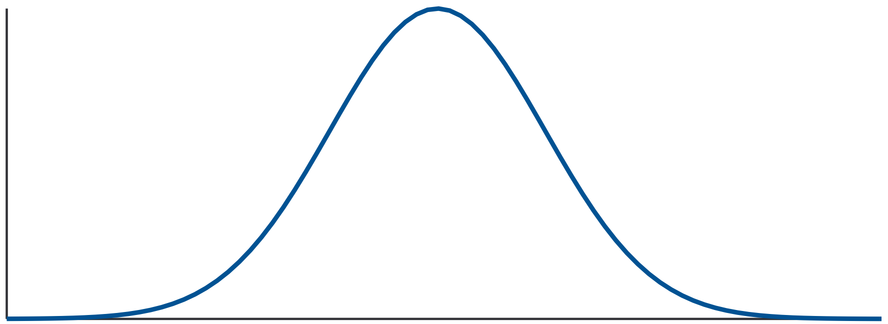
$\therefore x_{90} = \mu + \sigma z_{90}$

Highway Speed: Fastest 10% of Drivers

Example 3.15

Context: Vehicle speeds on a 100 km/h highway are approximately $N(117, 8^2)$ km/h.
Find: Above what speed are the fastest 10% of drivers?

We want x_{90}: $P(X \leq x_{90}) = 0.90$



- Step 1: Find z_{90}, the 90^{th} percentile of $N(0,1)$.
  We find from the table that $z_{90} = 1.28$
-

Step 2: Recall that $x_{90} = \mu + \sigma z_{90} = 117 + 8 \cdot (1.28)$
$\Rightarrow x_{90} = 127.2$.
$\therefore$ The 90^{th} percentile of X is 127.2 km/h.

Standard Normal Table

$z_{90} \doteq 1.28$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Three-Step Method: Find a Value Given a Proportion

Use this when you know a percentage (like "the top 10%") and want to find the cutoff value.

Inverse Normal Calculation

1. **State and sketch:** Draw the Normal curve and shade the given area. Express it as a **left-tail** cumulative proportion p.
2. **Use standard normal table backwards:**
   If $p > 0.5$: search the body of standard normal table for p and read off z (positive).
   If $p < 0.5$: search for $1 - p$ instead, read off the positive z, then negate it.
3. **Unstandardize:** Convert back to the original scale:

$$x = \mu + z\sigma$$

Highway Speed: Middle 90% of Traffic

Example 3.16

Context: Vehicle speeds follow $N(117, 8^2)$ km/h.

Find: The speeds x_L and x_U that bound the middle 90% of traffic (5% in each tail).

We want x_L and x_u s.t. $P(x_L \le X \le x_u) = 0.9$

and that this is the middle 90%.

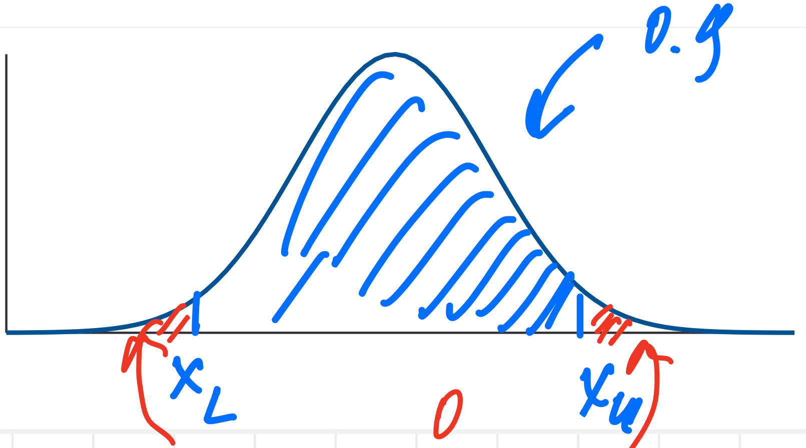
$$P(X \le x_L) = \frac{1}{2}\,0.1 = P(X \ge x_u)$$

$$= 0.05$$

Question: What percentile does x_u correspond to?

Note that the right tail prob at x_u is 0.05 $\Rightarrow$ $x_u = x_{95}$ … $x_L = x_{05}$

$x_{95} = 117 + 8(1.645) \doteq 130$, $x_L = 117 + 8(-1.645) = 104$



$P(X \le x_L) = P(X \ge x_u)$

Note that

$P(X \le x_L) + P(X \ge x_u)$

$= 1 - P(x_L \le X \le x_u)$

$= 1 - 0.9$

$= 0.1$

$\therefore$ The desired int. is $(104, 130)$

Some caveats (for the final exam and last assignment)

- If a question can be answered using the 68–95–99.7 rule, you don't need to use the standard normal table. You can just give an approximate answer based on the rule.

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- If a question can be answered using the 68–95–99.7 rule, you don't need to use the standard normal table. You can just give an approximate answer based on the rule.
- The normal table only provides values for z between 0 and 3.49. For z greater than 3.49, the area is so close to 1 that we can treat it as 1 for all practical purposes.
- Similarly, for $z < -3.49$, the area is so close to 0 that we can treat it as 0 for all practical purposes.

Some caveats (for the final exam and last assignment)

- If a question can be answered using the 68–95–99.7 rule, you don't need to use the standard normal table. You can just give an approximate answer based on the rule.
- The normal table only provides values for z between 0 and 3.49. For z greater than 3.49, the area is so close to 1 that we can treat it as 1 for all practical purposes.
- Similarly, for $z < -3.49$, the area is so close to 0 that we can treat it as 0 for all practical purposes.
- When looking up values in the standard normal table, if the exact value is not listed, choose the closest one. This will give you an approximation of the area.
- If two values are equally close (e.g. attempting to look up $\Phi(1.375)$), you can choose either one (e.g. choose $\Phi(1.37)$ or $\Phi(1.38)$). The difference in area will be very small, and it won't significantly affect your answer.

CHAPTER 3

Summary

Key ideas and formulas from this chapter

Chapter 3 Summary

Density Curves

- Area $= 1$, always ≥ 0
- Area under curve $=$ proportion
- Idealized model for data distributions

Normal Distribution

- $N(\mu, \sigma^2)$: symmetric, bell-shaped
- μ controls center, σ controls spread
- 68–95–99.7 rule for quick estimates

z-Scores & Calculations

- Standardize: $z = \frac{x-\mu}{\sigma}$
- Unstandardize: $x = \mu + z\sigma$
- $\Phi(z) = P(Z \leq z)$ (standard normal table, $z \geq 0$)
- Symmetry: $\Phi(-z) = 1 - \Phi(z)$
- Right tail: $1 - \Phi(z)$
- Between: $\Phi(b) - \Phi(a)$
- Always draw and shade first

PRACTICE

Problems

Applying the tools from Chapter 3

Practice Problem 1: Density Curves

Example 3.17

Context: The time a student spends waiting for a campus shuttle can be modelled by a density curve that is a straight line from $(0, 0.4)$ to $(5, 0)$ and is 0 elsewhere (shuttles run every 5 minutes).

(a) How do we verify this is a valid density curve?
(b) ~~What proportion of students wait less than 2 minutes?~~
(c) ~~What is the probability a student waits between 1 and 4 minutes?~~

Omit

Practice Problem 2: The 68–95–99.7 Rule

Example 3.18

Context: Resting heart rate among healthy adults follows $N(70, 10^2)$ bpm.

(a) What percentage of adults have a resting heart rate between 60 and 80 bpm?
(b) What percentage of adults have a resting heart rate above 80 bpm?
(c) What percentage of adults have a resting heart rate below 50 bpm?
(d) Above what heart rate are the highest 0.15% of adults?

Practice Problem 3: z-Scores

Example 3.19

Context: Testosterone levels among men follow $N(724, 221^2)$ ng/dL. Testosterone levels among women follow $N(33, 12^2)$ ng/dL.

(a) Find the z-scores for a man at 500, 800, and 1200 ng/dL.

(b) A woman at 45 ng/dL and a man at 1000 ng/dL. Who has more unusual testosterone?

(c) What testosterone level corresponds to a z-score of -2 for women?

Practice Problem 4: Forward Normal Calculations

Example 3.20

Context: Caffeine content in 16 oz specialty coffees follows $N(188, 36^2)$ mg.

(a) What proportion of cups contain less than 140 mg of caffeine?

(b) What proportion contain more than 250 mg?

(c) What proportion contain between 140 and 224 mg?

Practice Problem 5: Inverse Normal Calculations

Example 3.21

Context: Vehicle speeds on a 100 km/h highway follow $N(117, 82)$ km/h.

(a) Below what speed are the slowest 5% of drivers?

(b) What speed is at the 25th percentile?

(c) Find the speeds that bound the middle 50% of traffic.

Practice Problem 6: Screen Time

Example 3.22

Context: Daily smartphone screen time among college students is approximately Normally distributed with a mean of 7 hours and a standard deviation of 2.7 hours.

(a) What percentage of students have screen time between 9 and 12 hours per day?
(b) What is the probability a student has exactly 7 hours of screen time?
(c) How low would screen time need to be to fall in the lowest 25%?