Formul

Chapter 12

Introduction to Probability

Intended Learning Outcomes

- Define random phenomena, outcomes, and sample spaces
- Identify and describe events
- Apply the three axioms of probability
- Use the addition rule (disjoint and general)
- Apply the complement rule
- Use Venn diagrams to visualize probabilities
- Distinguish finite and continuous probability models
- Define and interpret random variables

PART 1

From Tables to Probability

Example 12.1: From Chapter 6 to Chapter 12

Recall the flu vaccine study (Example 6.7):

Group	**Developed Flu**	**No Flu**	**Total**
Vaccinated	26	3874	3900
Control	70	3830	3900
Total	96	7704	7800

In Chapter 6, you computed:

- Marginal proportion of flu: $\frac{96}{7800} = 0.0123$
- Conditional proportion of flu given vaccinated: $\frac{26}{3900} = 0.0067$

In Chapter 12 notation:

- $P(\text{Flu}) = 0.0123$ *(probability of flu in the whole study)*
- $P(\text{Flu} \mid \text{Vaccinated}) = 0.0067$ *(probability of flu given vaccinated)*

Why Do We Need Formal Rules?

With contingency tables, we could answer questions like:

- What proportion have the flu?
- What proportion are vaccinated *and* have the flu?
- What proportion are vaccinated *or* have the flu?

But what about more complex situations?

- What if we can't enumerate all outcomes in a table?
- What if outcomes happen in stages (first this, then that)?
- What if we need to combine information from multiple sources?

Key Point: Probability theory gives us **general rules** that work in any situation, not just contingency tables.

PART 2

Foundations of Probability

Random Phenomena and Outcomes

Random Phenomenon

A **random phenomenon** is any process whose result is uncertain before it occurs.

An **outcome** is a single possible result of a random phenomenon.

Outcome not yet determined:

- Flipping a coin
- Rolling a die
- Drawing a card

Outcome determined but unknown to us:

- A patient's disease status before testing
- Tomorrow's weather
- Whether a randomly selected person is left-handed

Chapter 6 Connection

When you randomly selected a person from a contingency table, you were observing a random phenomenon. The uncertainty about their group membership is what made proportions meaningful.

Sample Space

Sample Space

The **sample space** is the set of **all possible outcomes** of a random phenomenon. We denote it by S.

🧠 Notation

We write sets using curly braces: $\{\ \}$. The braces hold all possible outcomes without regard to order or duplicates.

$S = \{HH, TH, HT, TT\}$

Unordered list w/o duplicates

Quick examples:

- Flip a coin: $S = \{H, T\}$
- Roll a die: $S = \{1, 2, 3, 4, 5, 6\}$
- Pick a person from the flu study: $S = \{$all 7800 participants$\}$

E.g. $\{H, T\} = \{T, H\}$

Example 12.2: Writing Sample Spaces

Task: Write the sample space for each random phenomenon.

1. Flipping a coin
2. Rolling a six-sided die
3. Flipping a coin twice

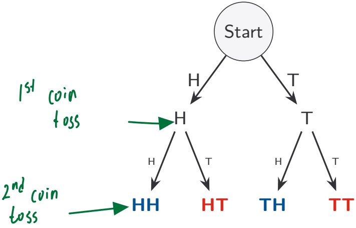
1) $S = \{H, T\} = \{T, H\}$

2) $S = \{1, 2, 3, 4, 5, 6\}$

Tree Diagrams: Flipping Two Coins

Context: Flip a coin twice. What are all possible outcomes?

Key Point: A **tree diagram** helps visualize all possible outcomes when a random process has multiple steps.



Sample space: $S = \{HH, HT, TH, TT\}$: 4 possible outcomes.

Events

Event

An **event** is a set of outcomes from a sample space.

We rarely care about every outcome individually. Instead, we ask about **events**:

- "At least one head" (when flipping coins)
- "Rolling an even number" (when rolling a die)
- "The person has the flu" (when sampling from the flu study)

Chapter 6 Connection

In contingency tables, each row or column defined an event. "Vaccinated" was an event; "Developed Flu" was an event.

Example 12.3: Writing a Sample Space

Context: Flip a coin three times. Each flip is independent.

1. List all possible outcomes (sequences of flips):
2. Let W = "At least two heads." List the outcomes in W:

Experiment: toss a coin three times.

1) Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

2) W = "at least two heads"
= {HHH, HHT, HTH, THH}.

Example 12.4: Identifying Events

Task: In every experiment, list the outcomes belonging to every event.

$S = \{H, T\}$

1. Flip a coin. Event $A =$ "Get heads"
2. Roll a die. Event $B =$ "Roll an even number"
3. Draw a card. Event $C =$ "Draw a face card"
4. Flip two coins. Event $D =$ "At least one tail"

1) $A = \{H\}$

4) $D = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}.$

2) $B = \{2, 4, 6\}$

3) There are 4 suits: hearts, spades, diamonds, clubs
Faces: Jack, Queen, King

J♣ Q♣ K♣ J♢ Q♢ K♢ J♠ Q♠ K♠ J♡ Q♡ K♡

$C = \{(s, f) :$ s is one of hearts, spades, diamonds or clubs, f is one of Jack, Queen and King $\}$

2-tuple

use all combinations according to the rule after the :

Last time

Sample space : all possible outcomes
Outcomes : results of the experiment
Events : collections of outcomes

Flip a coin twice

$S = \{HH, HT, TH, TT\}$

HH is an outcome

$A =$ "at least one H"

$= \{HH, HT, TH\}$.

Combining Events: "Or" and "And"

S = "rolling a 6-sided die"

$= \{1, 2, 3, 4, 5, 6\}$

Union and Intersection

A = "roll an even number"

B = "roll at least a 4"

- **Union** "A or B": All outcomes in A, in B, or in both.
- **Intersection** "A and B": Only outcomes in *both* A and B.

Intersection: A and B = "roll an even number" and "roll at least a 4" $= \{4, 6\}$

⚠ **Caution:** In mathematics, "or" is **inclusive**: it means at least one occurs, possibly both. This differs from everyday English where "or" often means "one or the other, but not both."

Union: A or B = "roll an even number" or "roll at least a 4" $= \{2, 4, 6, 5\}$

Notation:

- **Union:** $A \cup B$ or "A or B"
- **Intersection:** $A \cap B$ or "A and B"

Example: Union and Intersection with Cards

Context: Draw one card from a standard 52-card deck.

- Event $A =$ "Draw a heart": 13 outcomes
- Event $B =$ "Draw a face card (J, Q, K)": 12 outcomes

Hearts (13): 2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

Face cards (12): J♣ Q♣ K♣ J♢ Q♢ K♢ J♠ Q♠ K♠ J♡ Q♡ K♡

Calculate the number of outcomes in each event:

- $A \cap B$ ("heart **and** face card"):
- $A \cup B$ ("heart **or** face card"):

Intersection: "A and B" $= A \cap B = \{J♡, Q♡, K♡\}$

\# of outcomes $= |A \cap B| = 3$

Example: Union and Intersection with Cards

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- Event B = "Draw a face card (J, Q, K)": 12 outcomes

Hearts (13): 2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ J♡ Q♡ K♡ A♡

Face cards (12): J♣ Q♣ K♣ J♢ Q♢ K♢ J♠ Q♠ K♠ J♡ Q♡ K♡

Calculate the number of outcomes in each event:

- $A \cap B$ ("heart **and** face card"):
- $A \cup B$ ("heart **or** face card"):

A and not B

Union: "A or B" = $A \cup B = \{$ 2♡ 3♡ 4♡ 5♡ 6♡ 7♡ 8♡ 9♡ 10♡ A♡,
J♣ Q♣ K♣ J♢ Q♢ K♢ J♠ Q♠ K♠, J♡ Q♡ K♡ $\}$

B and not A

A and B

Example 12.5: Combining Events

Context: Pavlov has two dogs. Each dog is male (M) or female (F), all combinations equally likely.

1. Sample space S
2. A_1: "First dog is male"
3. A_2: "Second dog is male"
4. "At least one male"
5. "Both male"

1) $S = \{MM, MF, FM, FF\},$

2) $A_1 = \{MM, MF\}$

3) $A_2 = \{MM, FM\}$

4) "A_1 or A_2" $= A_1 \cup A_2 = \{MM, MF, FM\}$

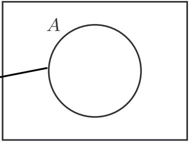
5) "Both male" $= \{MM\} = A_1 \cap A_2$

Venn Diagrams: A Visual Tool

Venn Diagram

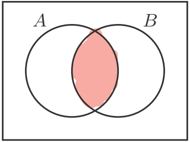
A **Venn diagram** represents events as regions inside a rectangle (the sample space). Overlapping regions show intersections.

Closed curves are events

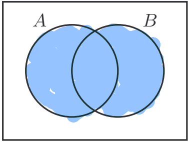


Event A

Rectangle = sample space



A and B

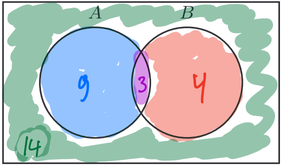


A or B

Example 12.12: Working with Venn Diagrams

Context: In a class of 30 students, 12 like apples (A), 7 like bananas (B), and 3 like both.

1. How many like only apples?
2. How many like only bananas?
3. How many like neither?



$S :=$ "all students"

$|A| =$ size of event $= \#$ outcomes

1) "A and not B" $= |A| - |A \cap B| = 12 - 3 = 9$

2) "B and not A" $= |B| - |A \cap B| = 7 - 3 = 4$

3) "not A and not B" $= |S| - (|A| + |B \text{ and not } A|)$ $= 30 - (12 + 4) = 14$

$|B| + |A \text{ and not } B|$

PART 3

Rules of Probability

What Is Probability?

Probability

The **probability** of an event is a number between 0 and 1 that measures how likely the event is to occur.

Three paradigms of probability:

1. **Classical:** Count equally likely outcomes.
   Example: Fair die: each face has probability $\frac{1}{6}$.
2. **Empirical:** Use long-run relative frequencies.
   Example: Track 1000 coin flips; proportion of heads ≈ 0.5.
3. **Subjective:** Based on personal judgment.
   Example: "I think there's a 70% chance it rains tomorrow."

Chapter 6 Connection

When you computed proportions from contingency tables, you were using the **empirical** approach: relative frequencies from observed data.

The Three Axioms of Probability

Axioms of Probability

A1: For any event A: $0 \leq P(A) \leq 1$. (nonnegative and boundedness)
A2: The probability of the entire sample space is 1: $P(S) = 1$. ("something always happens")
A3: If A and B are **mutually exclusive** (cannot both occur), then

$$P(A \text{ or } B) = P(A) + P(B)$$

Aside: A3 is sometimes referred to by "σ-additivity"

Example 12.6: Classical Probability

Def^n

~~**Key Point**~~**:** When outcomes are equally likely:

$$P(\text{event}) = \frac{\text{number of outcomes in event}}{\text{total number of outcomes}}$$

Calculate:

- $P(\text{rolling a 4})$ ↗ $\{4\}$

  $\frac{1}{6}$

- $P(\text{rolling an even number})$ ↗ $\{2,4,6\}$

  $= \frac{3}{6}$

- $P(\text{rolling greater than 4})$ ↗ $\{5, 6\}$

  $\frac{2}{6}$

Example 12.7: When Outcomes Are Not Equally Likely

Context: A novelty die is weighted so that 6 appears twice as often as any other number. All other faces are equally likely.

1. Find $P(6)$:
2. Find $P(\text{even number})$:
3. Find $P(\text{not a }6)$:

2) $= p_2 + p_4 + p_6 = 2p_6 = \frac{4}{7}$

3) $= 1 - p_6 = \frac{5}{7}$

Not classical probability

1) Let p_i = Probability of rolling an i, for $i = 1, 2, 3, 4, 5, 6$

Goal: Find p_6.

As probability satisfies the three axioms A1 - A3, we know that

Addition Rule

or σ-additivity

① $P(S) = 1$

② $0 \le p_i \le 1$

③ $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$.

$\frac{1}{2}p_6 + \frac{1}{2}p_6 + \frac{1}{2}p_6 + \frac{1}{2}p_6 + \frac{1}{2}p_6 + p_6 = 1$

We know that $p_6 = 2p_1 = 2p_2 = 2p_3 = \cdots = 2p_5$

$\Rightarrow \frac{1}{2}p_6 = p_1 = p_2 = \cdots = p_5$

$\frac{7}{2}p_6 = 1 \Rightarrow p_6 = \frac{2}{7} \Rightarrow p_i = \frac{1}{7}$, for $i = 1, \ldots, 5$

Which paradigm do we use?

Which paradigm do we use in practice?

- **Empirical:** When we have data (contingency tables, surveys, experiments), we use observed proportions.
- **Classical:** When outcomes are equally likely (dice, cards, lotteries), we count outcomes.

In our course, we will only use the empirical and classical paradigms.

Convention

When no information suggests otherwise, assume equally likely outcomes.

Example 12.8: Applying the Axioms

Context: A lottery draws a three-digit number from 000 to 999. Each is equally likely.

- Total outcomes: 1000
- Probability of each outcome: $\frac{1}{1000}$

Find:

1. P(all three digits equal)
2. $P(\text{number} < 500)$
3. P(number ends in 7)

1) $\overset{A}{\overset{\shortparallel}{\text{"all three digits equal"}}} = \{000, 111, 222, \overset{\text{"and so on"}}{\cdots}, 888, 999\} \leftarrow$ 10 outcomes

$$\Rightarrow P(A) = \frac{10}{1000} = \frac{1}{100}$$

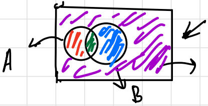
2) $\underset{B}{\underset{\shortparallel}{\text{"number} < 500\text{"}}} = \{000, 001, 002, \ldots, 498, 499\}$

$$\Rightarrow |B| = 500$$

$$\Rightarrow P(B) = \frac{500}{1000} = \frac{1}{2} = 0.5$$

Last Time

- Probability
  - $P(S) = 1$
  - $0 \leq P(A) \leq 1$
  - If A and B are disjoint or mutually exclusive,
    $P(A \text{ or } B) = P(A) + P(B)$
- Operations on Event: Let A and B be an events in our sample space
  - Union: $A \text{ or } B = A \cup B = \{\omega \mid \omega \text{ is outcome in } A \text{ or in } B\}$
  - Intersection: $A \text{ and } B = A \cap B = \{\omega \mid \omega \text{ is an outcome in both } A \text{ and } B\}$
- Venn diagrams



Sample space

we write the counts or probabilities in the disjoint regions

- Classical probability

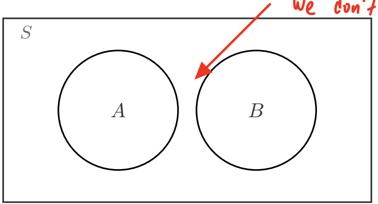
$$P(A) := \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } S}$$

Mutually Exclusive Events

Mutually Exclusive (Disjoint)

Two events are **mutually exclusive** if they cannot both occur. They share no outcomes.

We don't draw an overlap when the events are mutually exclusive or disjoint.



Example: Rolling a die: "Roll a 2" and "Roll a 5" are mutually exclusive.

Example 12.9: Checking Mutual Exclusivity

Context: Roll a fair six-sided die. Determine if each pair is mutually exclusive.

1. $A =$ "Roll a 2", $B =$ "Roll a 5"

2. $A =$ "Roll an even number", $B =$ "Roll greater than 3"

Addition Rule for Mutually Exclusive Events

Key Point: If A and B are mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B)$$

1. Roll a die. Find $P(1 \text{ or } 6)$.
2. Draw a card. Find $P(\text{heart or spade})$.

1) $P(1 \text{ or } 6) = P(1) + P(6) = \frac{2}{6}$, as A and B are disjoint.

$A =$ "1 is the upturned face"
$B =$ "6 is the upturned face"

"1 or 6" $= A \cup B$
and

Addition Rule

We'll come back to this in Chapter 13

Theorem: *Addition rule for multiple mutually exclusive events*

If $A_1, A_2, \dots, A_n$ are all mutually exclusive, then

$$P(A_1 \text{ or } \cdots \text{ or } A_n) = P(A_1) + \cdots + P(A_n)$$

Example 12.10: General Addition Rule Example

Context: A class has 100 students. 60 students own a laptop, 40 students own a tablet, and 20 students own both.

	Tablet	**No Tablet**	**Total**
Laptop	20	40	60
No Laptop	20	20	40
Total	40	60	100

Find: $P(\text{Laptop or Tablet})$

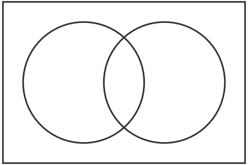
We'll come back to this in Chapter 13

The General Addition Rule

Addition Rule (General)

For **any** two events A and B:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



🔑 **Key Point:** We subtract $P(A \text{ and } B)$ because when A and B overlap, adding $P(A) + P(B)$ counts the overlap **twice**.

Example 12.13: Movie Nights

Context: At a movie night with 300 attendees:

- 120 watched a comedy (A)
- 150 watched a drama (B)
- 60 watched both a comedy and a drama ($A \cap B$)

Find: What is the probability that a randomly selected attendee watched **a comedy or a drama**?

The Complement of an Event

Complement

The **complement** of event A, written $\overline{A}$ or A^c, is the event that A does **not** occur.

Find the complement for each event:

1. $A =$ "roll a 2" on a die.
2. $B =$ "get at least one head in 3 coin flips".
3. $C =$ "draw a heart from a deck".

1) $\overline{A}$ = "not rolling a 2" = $\{1, 3, 4, 5, 6\}$.

2) $\overline{B}$ = " there are no heads in 3 coin tosses"
= $\{TTT\}$

The Complement Rule

A'

Complement Rule

For any event A,

$$P(\overline{A}) = 1 - P(A)$$

Example: Tossing a coin 3 times. Event A = "At least one head in 3 flips"

Find $P(A)$

- Note that $A = \{TTT\}$. $\Rightarrow$ $P(A) = \dfrac{\#\text{ of outcomes in } A}{\#\text{ of outcomes in } S}$

$$= \frac{1}{8}$$

$$\Rightarrow P(A) = 1 - P(\overline{A}) \text{, by complement rule}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

Example 12.11: Using the Complement Rule

Context: Roll a fair six-sided die.

1. $A =$ rolling a 2. Find $P(\overline{A})$:
2. $B =$ rolling less than 4. Find $P(\overline{B})$:

Example 12.11: Using the Complement Rule

Context: Roll a fair six-sided die.

1. Let $C =$ "roll an even number" and $D =$ "roll greater than 3". Find $P(C \cup D)$ and $P(\overline{C \cup D})$:

PART 4

Probability Models & Random Variables

Two Types of Probability Models

Finite Probability Model

A model whose sample space has a **finite number** of outcomes. We can list every outcome and its probability.

Continuous Probability Model

A model whose sample space is **continuous**: outcomes can take any value in an interval. We describe probabilities using a density curve.

🧠 Quick Test

Ask: “Can I list every possible outcome?”

- **Yes** → finite (e.g., number of children, die roll)
- **No** → continuous (e.g., height, time, temperature)

Finite or Continuous?

Task: Classify each scenario.

- Height of a person: Continuous
- Number of cars sold: Finite
- Time to run a mile: Continuous
- Number of siblings: Finite
- Goals in a hockey game: Finite
- Body temperature: Continuous
- Eggs in a carton: Finite
- Gas remaining in tank: Continuous

Remark: In this course, we assume that models which are not continuous must be finite. This is not true in general. There are many experiments where there are infinitely many outcomes with nonzero probability. A more accurate delineation would be discrete vs. continuous.

Example 12.14: A Finite Probability Model

Correction: No plus sign

Context: "How many cups of coffee have you had today?"

Cups (X)	0	1	2	3	4	5~~+~~
Probability	0.40	0.22	0.15	0.10	0.07	0.06

1. Verify this is a valid probability model and find
2. $P(X < 4) =$ "fewer than 4 cups"
3. $P(X \geq 1) =$ "at least one cup"

Exercise: Completing a Probability Model

Context: A discrete random variable Y has the following distribution. One probability is missing.

Value (Y)	10	20	30	40
Probability	0.15	0.30	**?**	0.10

Find the probability that $Y = 30$.

Random Variables

Random Variable

A **random variable** assigns a **numerical value** to each outcome of a random phenomenon. The **probability distribution** describes which values are possible and how likely each is.

Examples:

- X = number shown when rolling a die
- Y = number of heads in 10 coin flips
- Z = height (in cm) of a randomly selected student

experiment

$Y(HHHTHHHTTH) = 7$

Caution: Random variables must be **numerical**. "Favourite colour" is not a random variable (it's categorical). But we could define $X = 1$ if blue, $X = 0$ otherwise.

Example 12.15: A Discrete Random Variable

Context: Let X = number of countries visited by a randomly selected DS1000 student.

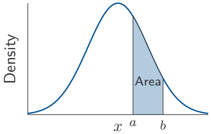
Countries	0	1	2	3	4	5	$\cdots$
Count	3	6	18	33	25	24	$\cdots$
Proportion	0.015	0.029	0.088	0.162	0.123	0.118	$\cdots$

1. $P(X = 3)$:
2. $P(X \leq 2)$:
3. $P(X > 2)$:

Continuous Random Variables

Some facts about continuous random variables:

- Continuous variables take values in an interval (e.g., height, time).
- Probabilities are represented by **area under a density curve**.
- The probability of any single exact outcome is 0.
- We calculate probabilities for intervals: $P(a < X < b)$.



$P(a < X < b)$

We'll come back to continuous random variables after Chapter 13.

CHAPTER 12

Summary

Chapter 12 Summary

Foundations

- **Random phenomenon:** uncertain outcome
- **Sample space S:** all possible outcomes
- **Event:** a set of outcomes
- **Union (or):** at least one occurs
- **Intersection (and):** both occur
- **Complement:** event does not occur

Models

- **Finite:** list outcomes in a table
- **Continuous:** use density curves
- **Random variable:** numerical outcome of a random phenomenon

Rules

- **A1:** $0 \leq P(A) \leq 1$
- **A2:** $P(S) = 1$
- **A3 (disjoint):** $P(A \bigcup B) = P(A) + P(B)$
- **General addition:** $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$
- **Complement:** $P(\overline{A}) = 1 - P(A)$

Looking Back and Looking Ahead

Key Point: Chapter 6 gave you **intuition** for probability through contingency tables. Chapter 12 gives you **formal rules** that work in any situation.

You Already Know:

- Proportions from tables
- Conditional proportions $P(A \mid B)$
- Why we subtract overlap
- Complement ("not A")

What's New:

- Formal axioms
- Sample space notation
- Tree diagrams
- Random variable notation

Coming Next

Chapter 13 will focus on **conditional probability** in depth, the formal version of the conditional proportions you computed in Chapter 6.

CHAPTER 12

Practice Problems

Example 12.17: Sample Spaces & Events

Context: A bag contains 3 marbles: Red (R), Green (G), and Blue (B). You draw two marbles one at a time **without replacement**.

Find:

1. List the sample space S. (Hint: use a tree diagram.)
2. Let $A =$ "Red is drawn at some point." List the outcomes in A.
3. Let $B =$ "Green is drawn first." List the outcomes in B.
4. Find $P(A)$, $P(B)$, and $P(A \cap B)$.
5. Are A and B mutually exclusive? Justify.

Example 12.18: Venn Diagrams

Context: A survey of 200 university students found:

- 90 use Instagram (I)
- 110 use TikTok (T)
- 50 use both

Find:

1. Fill in all four regions of a Venn diagram with counts.
2. $P(I \cup T)$ (uses Instagram or TikTok)
3. $P(\overline{I \cup T})$ (uses neither)
4. $P($uses exactly one of the two platforms$)$

Example 12.19: Finite Probability Models

Context: Let $X =$ number of pets owned by a randomly selected student. The probability distribution is:

Pets (X)	0	1	2	3	4+
Probability	0.18	0.35	0.25	**?**	0.07

Find:

1. Find $P(X = 3)$.
2. Find $P(X \geq 2)$.
3. Find $P(X < 2)$.
4. Is X a finite or continuous random variable? Justify.

Example 12.20: Weighted Outcomes

Context: A spinner has four sections: Red, Blue, Green, and Yellow. Red is twice as likely as Blue. Green and Yellow are each equally likely as Blue.

Find:

1. Find the probability of landing on each colour.
2. $P(\text{Red or Green})$. Are these events mutually exclusive?
3. $P(\text{not Yellow})$.

Example 12.21: Languages

Context: At a company, $P(\text{speaks French}) = 0.30$, $P(\text{speaks Spanish}) = 0.25$, and $P(\text{speaks French or Spanish}) = 0.45$.

Find:

1. $P(\text{speaks both French and Spanish})$
2. $P(\text{speaks neither French nor Spanish})$
3. Are "speaks French" and "speaks Spanish" mutually exclusive? Justify.

Example 12.22: Another password

Context: A password consists of one letter (A, B, or C) followed by one digit (1, 2, 3, or 4). All passwords are equally likely.

Find:

1. How many outcomes are in the sample space?
2. P(password starts with A)
3. P(password contains an even digit)
4. P(starts with A **or** contains an even digit)
5. P(does not start with A **and** contains an odd digit)

Example 12.23: More events

Context: For events A and B: $P(A) = 0.6$, $P(B) = 0.5$, and $P(\overline{A \cup B}) = 0.1$.

Find:

1. $P(A \cup B)$
2. $P(A \cap B)$
3. $P(\text{exactly one of } A \text{ or } B)$
4. Are A and B mutually exclusive? Justify.