

DS 1000B: Chapter 12 Review

Introduction to Probability

PART 1

The Language of Probability

Key vocabulary and concepts

Random Phenomena and Sample Spaces

Random Phenomenon

A process whose result is uncertain before it occurs. Each possible result is called an **outcome**.

Notation: ω or s .

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The set of **all possible outcomes** of a random phenomenon.

Notation: S or Ω .

Quick example:

- Flip a coin twice: $S = \{HH, HT, TH, TT\}$
- Measure someone's height: $S = \{x : x > 0\} = \{ \text{all positive real numbers} \}$.

Events

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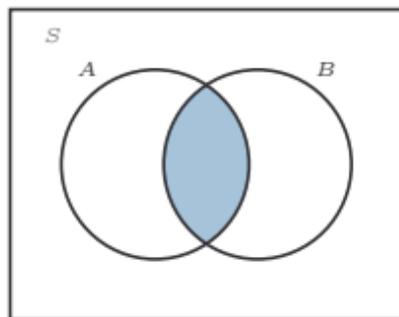
- “At least one head” (when flipping coins): $\{HH, HT, TH\}$
- “Roll an even number” (when rolling a die): $\{2, 4, 6\}$

Combining Events

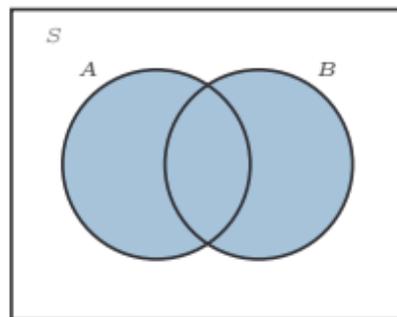
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$A \cap B$	"A and B"	Both A and B occur
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$A \cap B$: AND



$A \cup B$: OR

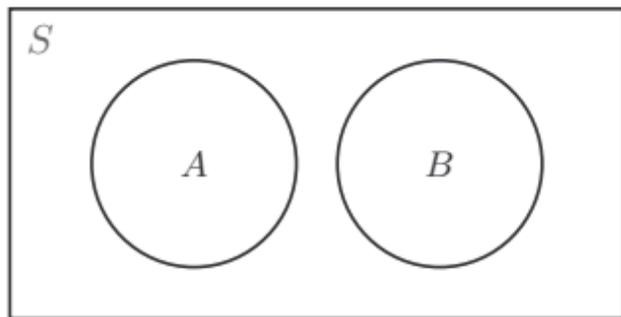
Venn diagrams
↙

⚠ Caution: "Or" in mathematics is **inclusive**: at least one, possibly both.

Mutually Exclusive Events

Mutually Exclusive (Disjoint)

Two events are **mutually exclusive** if they share no outcomes: $A \cap B = \emptyset$.

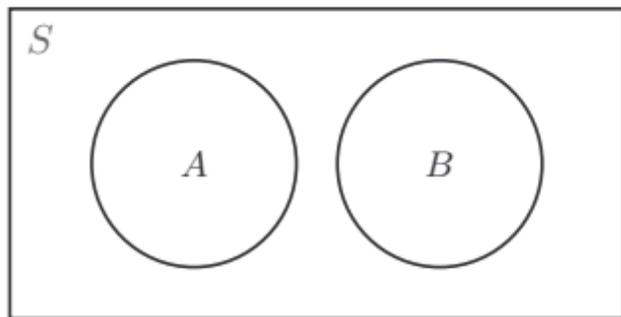


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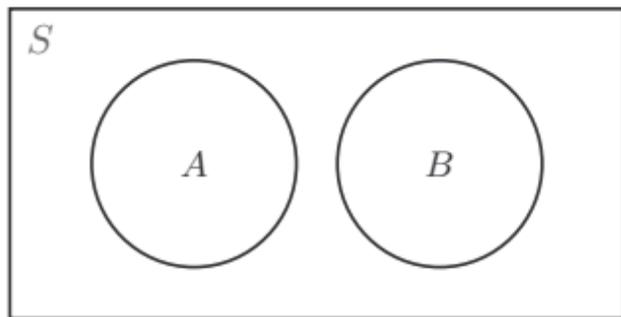
- $A =$ "roll a 2", $B =$ "roll a 5"

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Mutually exclusive

Quick example: Roll a die.

- $A =$ "roll a 2", $B =$ "roll a 5"

Mutually exclusive

- $A =$ "even", $B =$ "greater than 3"

Not mutually exclusive

4 and 6 are common to both

PART 2

The Rules of Probability

The three axioms and their consequences

The Three Axioms of Probability

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Axiom 1. For any event A : $0 \leq P(A) \leq 1$.

Axiom 2. $P(S) = 1$.

Axiom 3. If A and B are mutually exclusive:

$$P(A \cup B) = P(A) + P(B)$$

What Do the Axioms Mean?

The axioms seem simple, but they are powerful.

- **Axiom 1** says probabilities live on a scale from 0 (impossible) to 1 (certain).
No event can have a probability of -0.3 or 1.5 .

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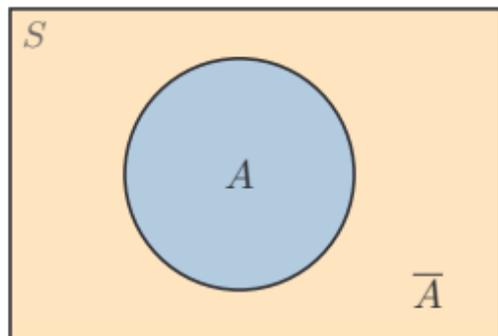
- **Axiom 1** says probabilities live on a scale from 0 (impossible) to 1 (certain).
No event can have a probability of -0.3 or 1.5 .
- **Axiom 2** says that an outcome in the sample space must occur.
- **Axiom 3** says for non-overlapping events, the probability of “one or the other” is the sum of the individual probabilities. This extends to any number of disjoint events:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

The Complement Rule

Complement

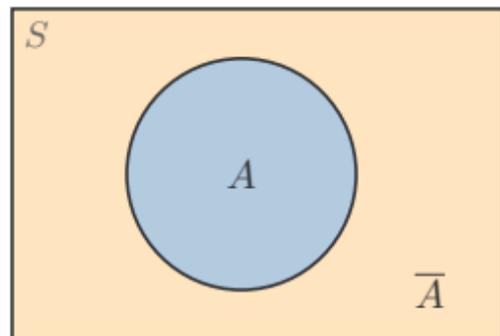
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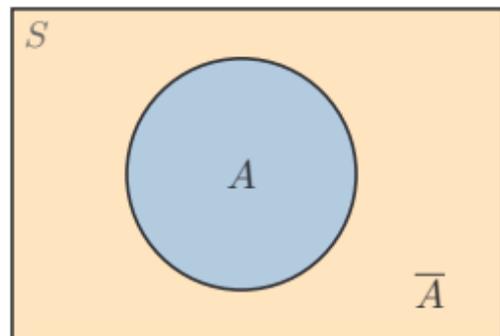
Since A and \bar{A} are disjoint and together make up all of S , the axioms give:

$$P(\bar{A}) = 1 - P(A)$$

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 **Key Point:** Use the complement when it is easier to calculate what you do **not** want.

The General Addition Rule

Axiom 3 only works when events are disjoint. What if they overlap?

General Addition Rule

For **any** two events A and B :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

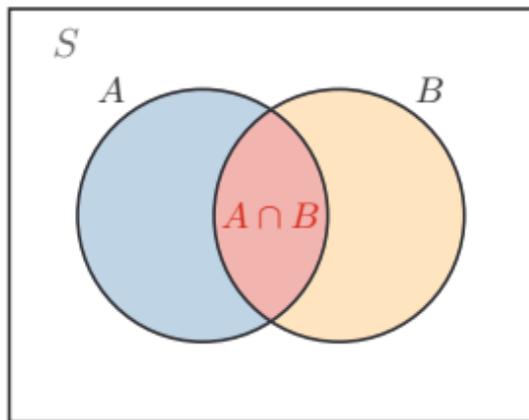
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We subtract $P(A \cap B)$ because adding $P(A) + P(B)$ counts the overlap **twice**.

When A and B are disjoint, $P(A \cap B) = 0$, so this reduces to Axiom 3.

Example: Movie Nights

Context: At a movie night with 300 attendees:

- 120 watched a comedy
- 150 watched a drama
- 60 watched both

Find: probability of a randomly selected attendee watching a comedy or a drama.

$A :=$ "comedy" $B :=$ "drama" we want $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A) = \frac{\text{\# of inds who watched a comedy}}{\text{\# of individuals}} = \frac{120}{300}$$

$$P(B) = \frac{150}{300}$$

$$P(A \cap B) = \frac{60}{300}$$

$$\Rightarrow P(A \cup B) = \frac{120}{300} + \frac{150}{300} - \frac{60}{300} = \frac{210}{300}$$

Example: Laptops and Tablets

Context: A class has 100 students. 60 own a laptop, 40 own a tablet, and 20 own both.

		B	\bar{B}	
		Tablet	No Tablet	Total
A	Laptop	20	40	60
\bar{A}	No Laptop	20	20	40
	Total	40	60	100

Find: probability of a randomly selected student having a laptop or a tablet.

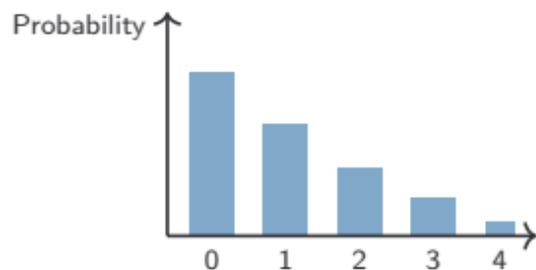
$$\begin{aligned} \text{Let } A &= \text{"laptop"}, B = \text{"tablet"} \\ \text{Then } P(A \cup B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= \frac{20+20}{100} + \frac{40+20}{100} - \frac{20}{100} \\ &= \frac{80}{100} \end{aligned}$$

PART 3

From Rules to Random Variables

Why probability distributions must sum to 1

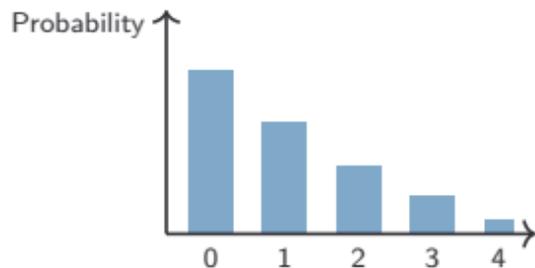
Finite Probability Models



Finite Probability Model

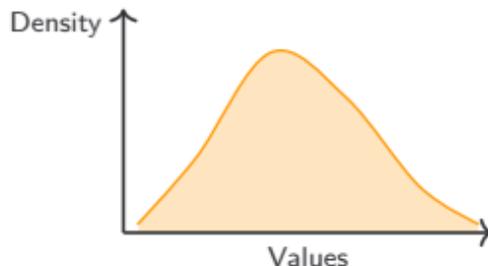
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Continuous Probability Model

- Values span an interval (e.g., height, time).
- Probabilities are **areas** under a density curve.
- Total area under the curve equals 1.
- Ask: does the variable take values on a continuous scale?

🔑 Discrete Random Variable

A finite probability model is part of a larger family of probability models known as **discrete** probability models.

In these models, the outcomes take on a value which is **countable**, meaning it is either finite or can be put into a one-to-one correspondence with the natural numbers (countably infinite list like $0, 1, 2, 3, \dots$).

Discrete Probability Models (Preview)

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Common examples you may encounter in future courses include the **binomial** and **Poisson** distributions.

What Is a Random Variable?

Random Variable

A **random variable** assigns a **numerical value** to each outcome of a random phenomenon. We use capital letters: X , Y , Z .

Quick example: Flip a coin twice. Let X = number of heads.

Outcome	Value of X	Event
TT	0	$\{X = 0\}$
HT	1	$\{X = 1\}$
TH	1	
HH	2	$\{X = 2\}$

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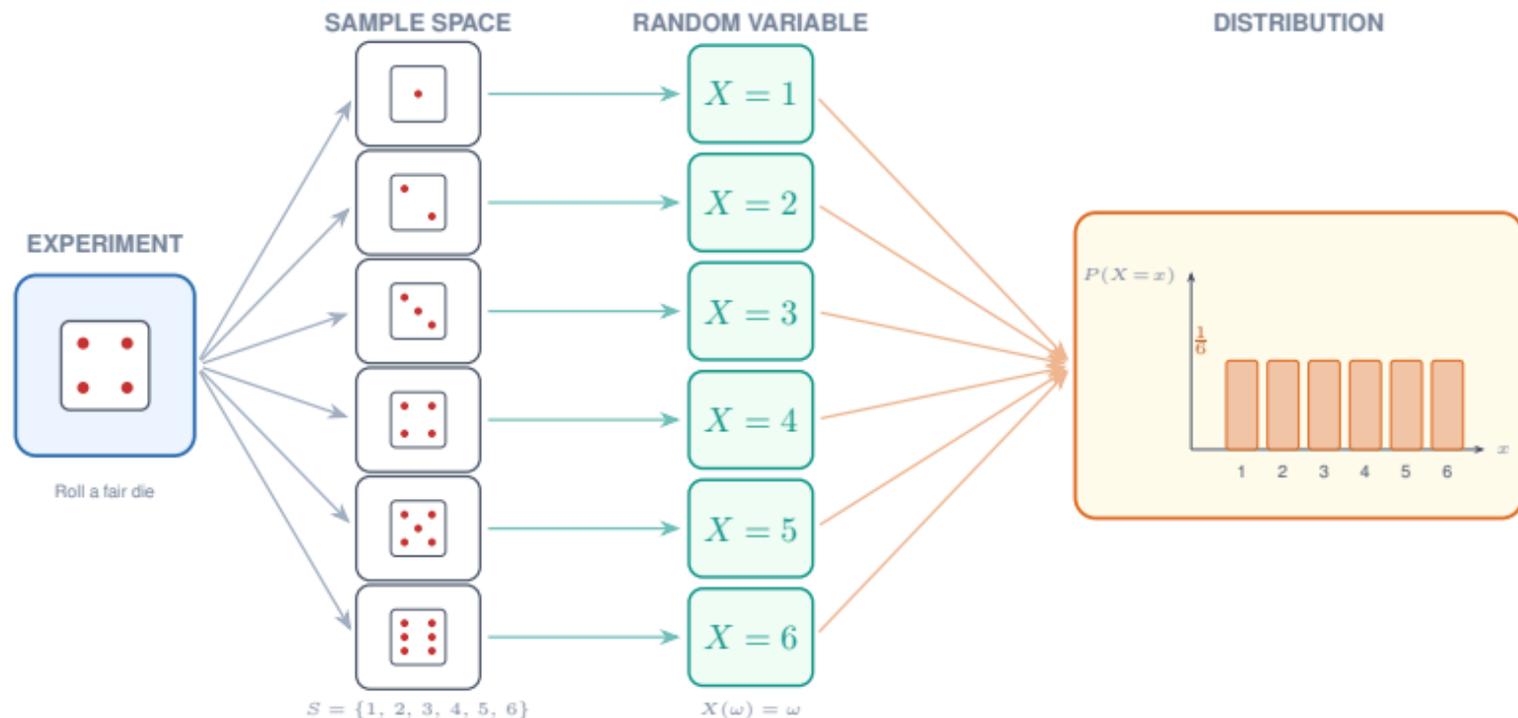
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Each value of X defines an **event**: a set of outcomes from the sample space. This is the key idea we will build on next.

Rolling a die



1 Perform the experiment

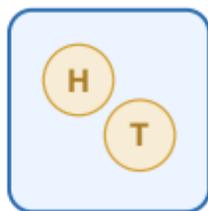
2 Observe one of the possible outcomes

3 Map outcome to a number via X

4 Repeating yields the distribution of X

Tossing two coins

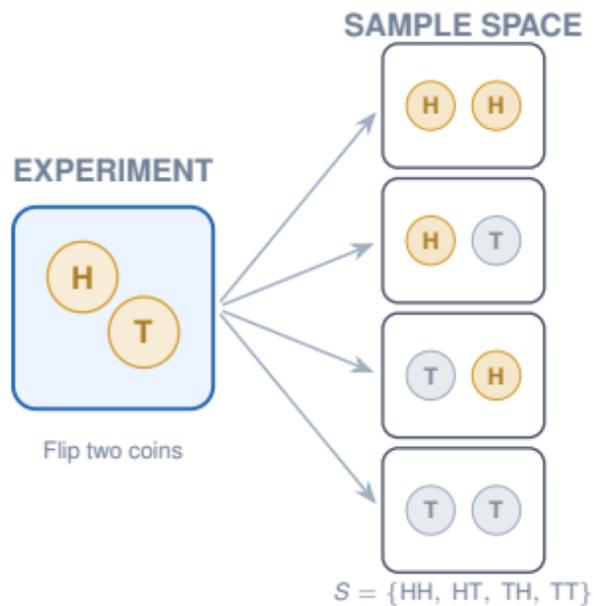
EXPERIMENT



Flip two coins

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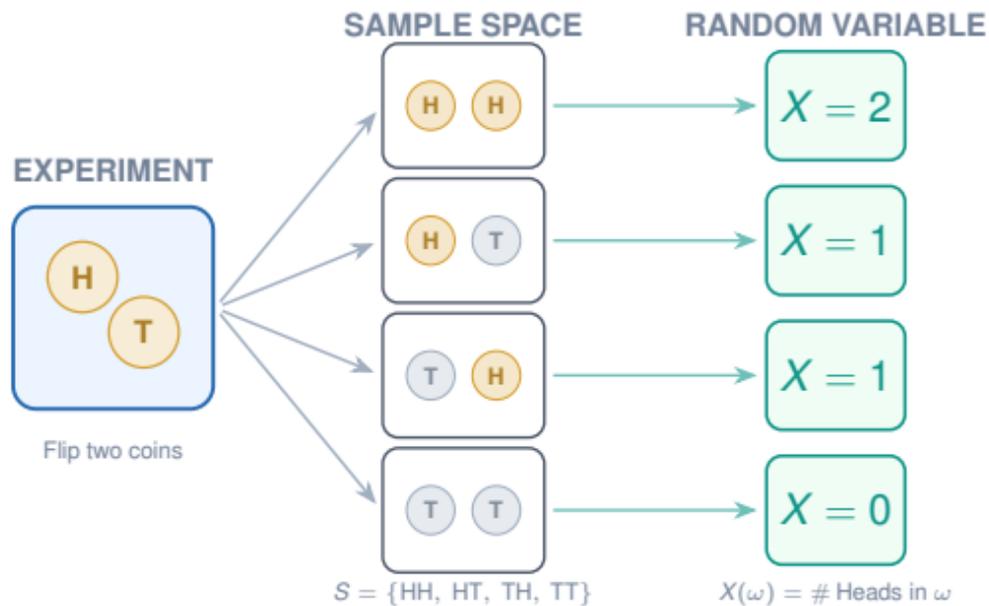
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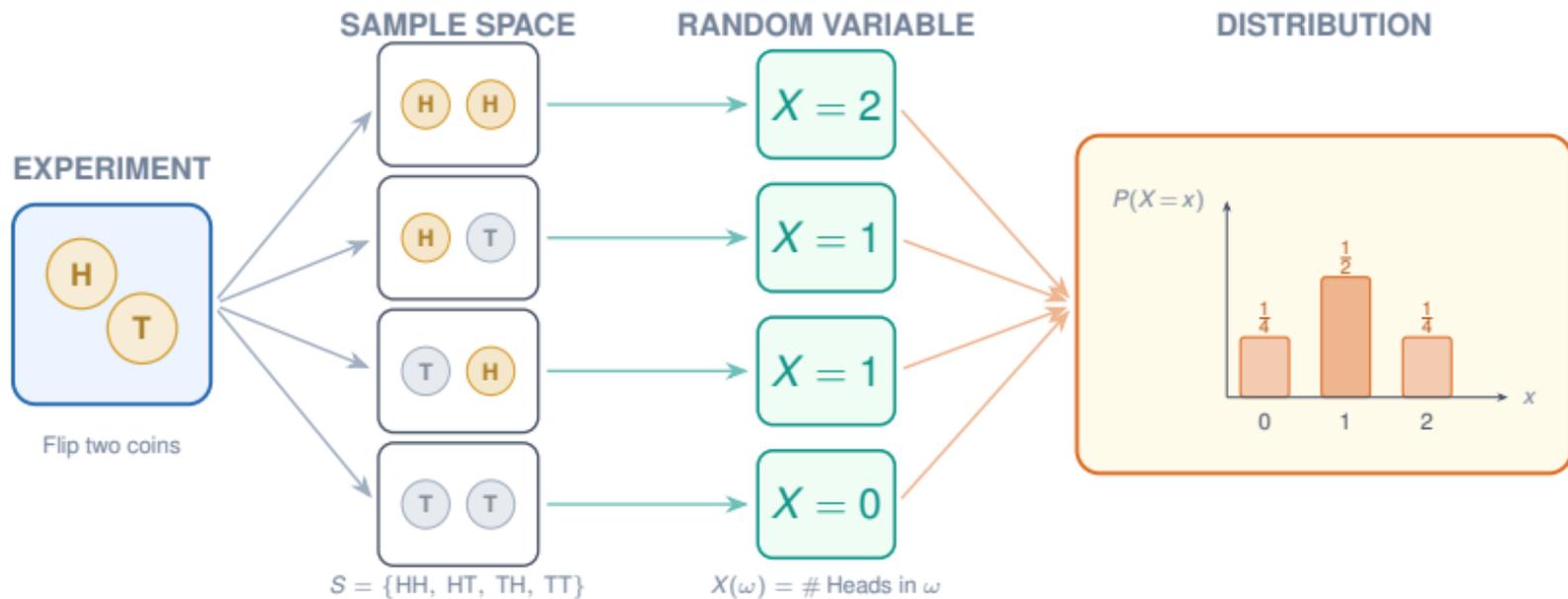


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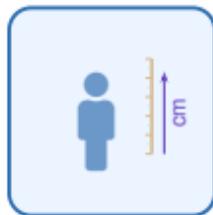
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Measuring height

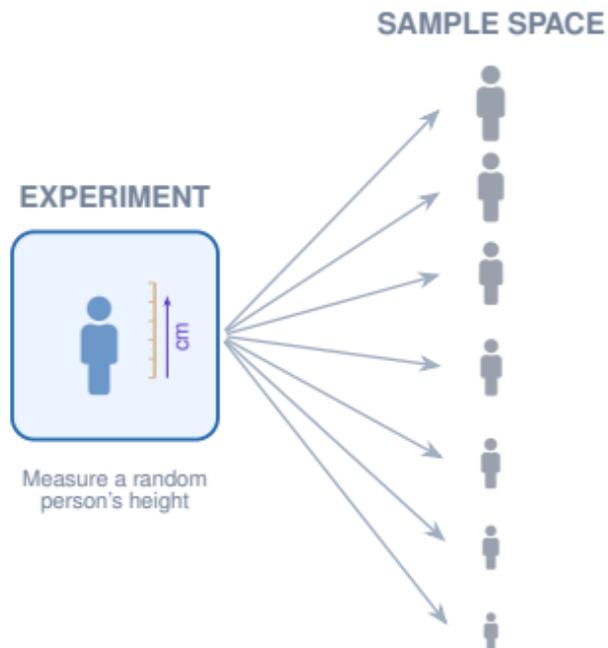
EXPERIMENT



Measure a random person's height

- 1 Perform the experiment

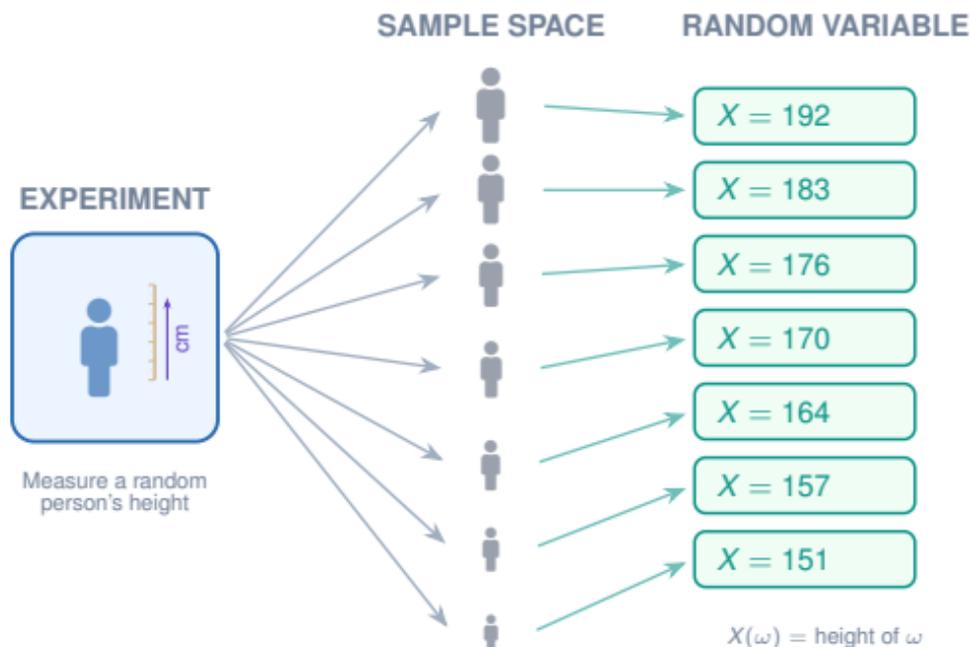
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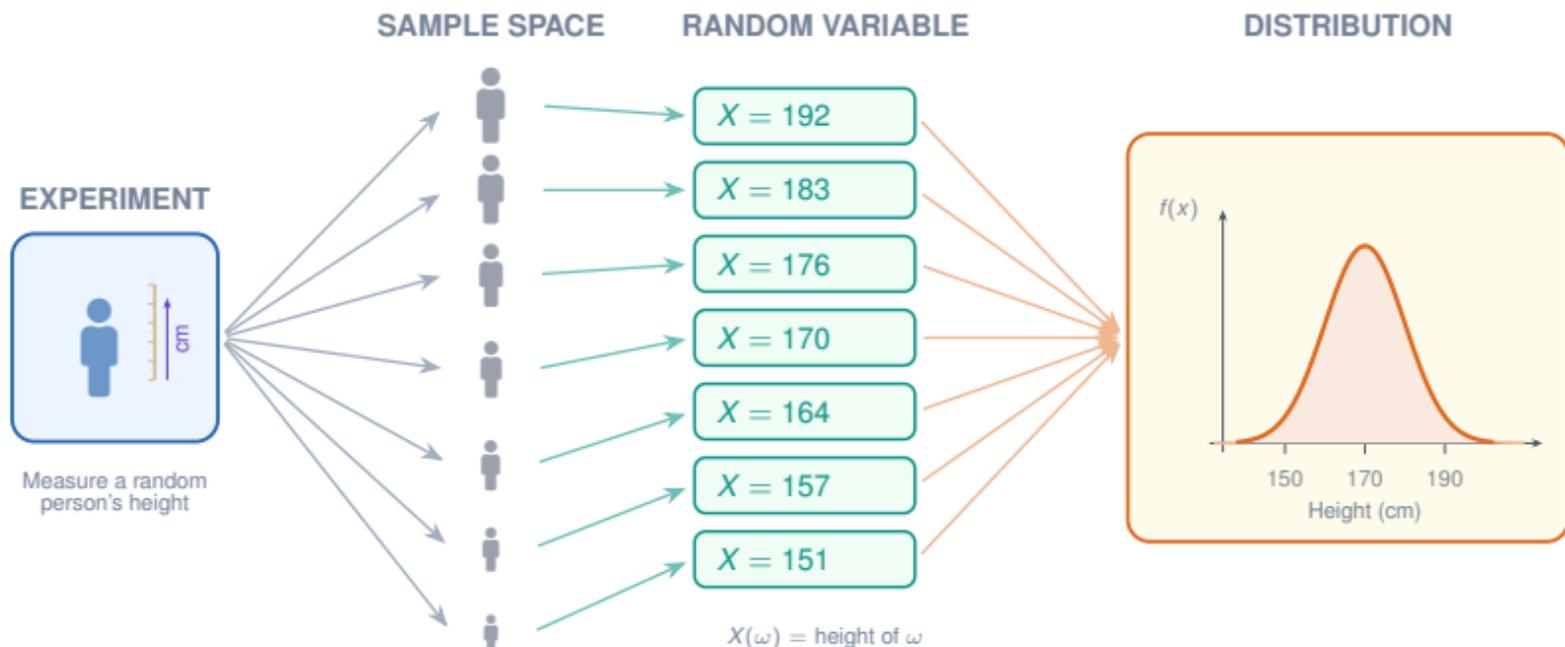


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Building a Probability Distribution

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Outcomes in event	TT	HT, TH	HH
$P(X)$	$1/4$	$2/4$	$1/4$

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Is this true in general?

Why Must Probabilities Sum to 1? Setup

Let's think carefully about what the events $\{X = 0\}$, $\{X = 1\}$, and $\{X = 2\}$ look like inside the sample space.



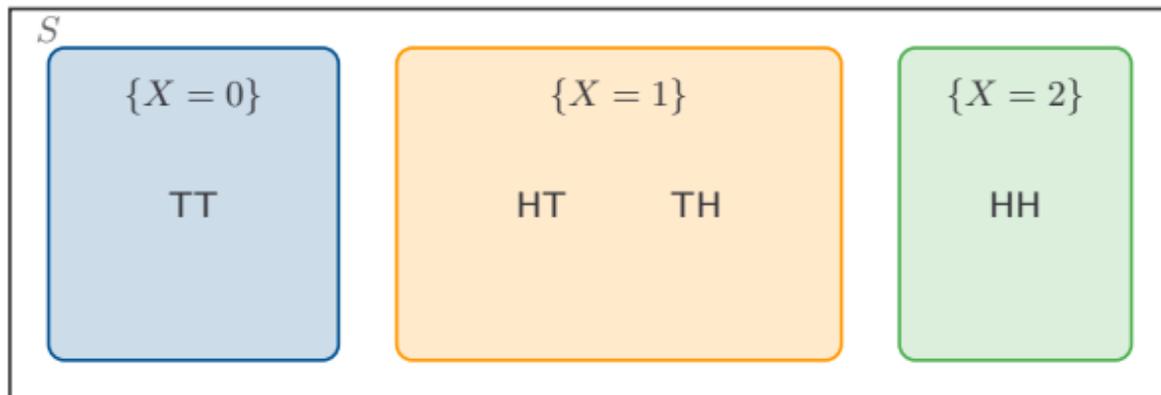
Two observations about these events:

1. They are **mutually exclusive**: no outcome belongs to two groups at once. (You cannot get 0 heads and 1 head simultaneously.)
2. They are **exhaustive**: every outcome in S belongs to exactly one group. (Every flip sequence produces *some* number of heads.)

In other words: $\{X = 0\} \cup \{X = 1\} \cup \{X = 2\} = S$.

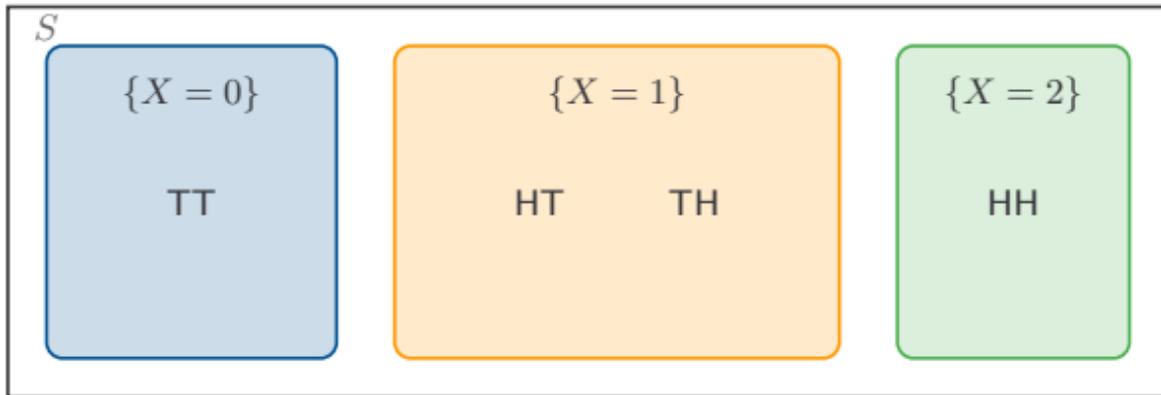
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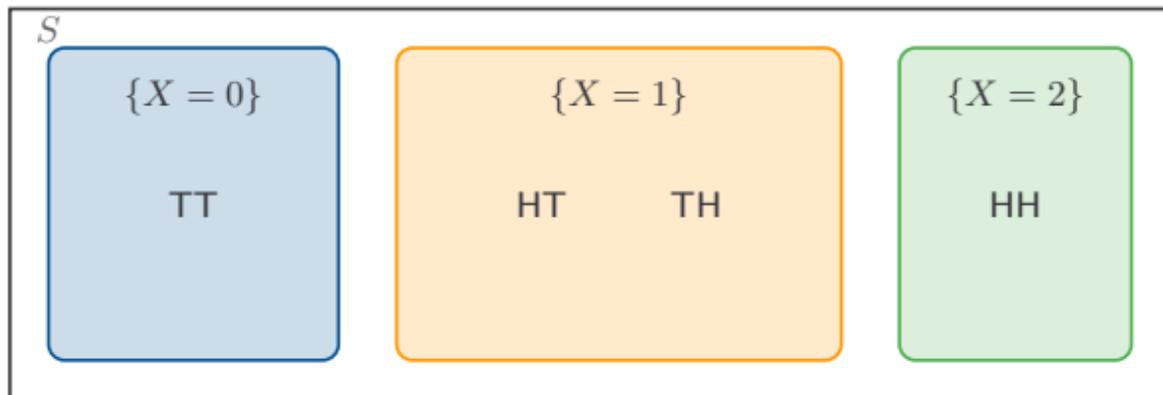


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In other words: $\{X = 0\} \cup \{X = 1\} \cup \{X = 2\} = S$.

$$\begin{aligned} P(S) \\ \parallel \\ P(X=0 \text{ or } X=1 \text{ or } X=2) \\ = P(X=0) + P(X=1) \\ + P(X=2) \\ = 1 \end{aligned}$$

Why Must Probabilities Sum to 1?

$$\{X=0\} \cup \{X=1\} \cup \{X=2\} = S$$

$$\Rightarrow P(\{X=0\} \cup \{X=1\} \cup \{X=2\}) = P(S)$$

$$\text{LHS} = P(\{X=0\}) + P(\{X=1\}) + P(\{X=2\})$$

$$\text{RHS} = 1.$$

The General Pattern

The same logic works for **any** finite random variable.

If X takes possible values x_1, x_2, \dots, x_k , then:

- The events $\{X = x_1\}, \{X = x_2\}, \dots, \{X = x_k\}$ are **mutually exclusive** (a random phenomenon produces exactly one value of X).
- Their union is S (every outcome maps to some value).

Therefore, by Axioms 2 and 3:

$$P(X = x_1) + P(X = x_2) + \dots + P(X = x_k) = 1$$

The Distribution of a Finite Random Variable

If X is a finite random variable with probability distribution given by:

Value	x_1	x_2	\cdots	x_k
Probability	p_1	p_2	\cdots	p_k

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- Each p_i satisfies $0 \leq p_i \leq 1$ (Axiom 1).

The Distribution of a Finite Random Variable

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Value	x_1	x_2	\cdots	x_k
Probability	p_1	p_2	\cdots	p_k

Then

- Each p_i satisfies $0 \leq p_i \leq 1$ (Axiom 1).
- The probabilities sum to 1 (Axioms 2 and 3):

$$p_1 + p_2 + \cdots + p_k = 1$$



How we check
if X is
a random variable

Example: Using the Rules with Random Variables

Context: X takes values 0, 1, and 2 with $P(X = 0) = 0.3$, $P(X = 1) = 0.5$, and $P(X = 2)$ unknown.

Find the following

(1) the missing probability $P(X = 2)$

$$\begin{aligned}P(X=2) &= 1 - P(X=0) - P(X=1) = 1 - 0.3 - 0.5 \\ &= 1 - 0.8 \\ &= 0.2\end{aligned}$$

(2) $P(1 \leq X \leq 2)$

$$\begin{aligned}P(1 \leq X \leq 2) &= P(X=1 \text{ or } X=2) \\ &= P(X=1) + P(X=2) \\ &= 0.5 + 0.2 = 0.7\end{aligned}$$

Worked Example: Coffee Survey

Context: X = number of cups of coffee a student had today.

Cups (X)	0	1	2	3	4	5
Probability	0.40	0.22	0.15	0.10	0.07	0.06

(a) What needs to be true for this to be a valid probability distribution?

- ① $1 \geq P(X = i) \geq 0$, for all $i = 0, 1, 2, 3, 4, 5$
- ② $P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) = 1$.

(b) Find $P(X \geq 1)$ = "at least one cup."

$$\begin{aligned} P(X \geq 1) &= 1 - P(\text{not } X \geq 1) \\ &= 1 - P(X < 1) = 1 - P(X=0) = 0.60 \end{aligned}$$

(c) Find $P(X < 4)$ = "fewer than 4 cups."

$$\begin{aligned} P(X < 4) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) = 0.87 \\ &= 1 - P(X \geq 4) = 1 - (P(X=4) + P(X=5)) = 0.87 \end{aligned}$$

CHAPTER 12

Summary

Key ideas from this review

Chapter 12 Summary

Language

- **Sample space S :** all possible outcomes
- **Event:** a set of outcomes
- **Union (\cup):** at least one occurs
- **Intersection (\cap):** both occur
- **Mutually exclusive:** $A \cap B = \emptyset$

Rules

- **Axiom 1:** $0 \leq P(A) \leq 1$
- **Axiom 2:** $P(S) = 1$
- **Axiom 3:** Disjoint \Rightarrow add probabilities
- **General addition:**
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Complement:** $P(\bar{A}) = 1 - P(A)$

Random Variables

- Assigns a numerical summary to each outcome (e.g., number of heads or height)
- The events $\{X = x_i\}$ are mutually exclusive and exhaustive
- The distribution of a finite random variable must satisfy: